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FINAL REPORT

A DYNAMIC ANALYSIS OF PIEZOELECTRIC STRAINED ELEMENTS

M. Cengiz DÖKMECİ

Istanbul Technical University
Teknik Üniversite, P.K.9
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TABLE OF CONTENTS

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CHAPTER 1

RECENT PROGRESS IN THE DYNAMIC APPLICATIONS OF PIEZOELECTRIC CRYSTALS

ABSTRACT

This chapter presents an updated review of open literature concerned with the dynamic applications of piezoelectric crystals. Representative current literature as well as previously surveyed literature that pertains to recent applications are reviewed for waves and vibrations in piezoelectric one-dimensional and two-dimensional elements. Experimental works and some numerical methods are briefly discussed, and future research needs are indicated.

1- INTRODUCTION

This review, the fifth in a series of surveys on the dynamic applications of piezoelectric crystals, deals with current open literature pertaining to waves and vibrations in piezoelectric structural elements. Accordingly, it supplements the earlier review papers [1-3] and should be considered in conjunction with them. The present compilation summarizes the rapid advancement of the subject due to the demand of both civil and military technology since 1983.

Theoretical as well as experimental investigations have been increasingly continuing for better design and better applications of piezoelectric elements since the last review article [3]. Comprehensive recent articles have discussed the design and applications of these elements [4-9], as have several monographs and books [10-21]. However, a detailed survey of design and application is excluded herein, as before.

The purpose of this review is to guide and to stimulate the reader through the pertinent literature that covers the most recent contributions to one-dimensional and two-dimensional piezoelectricity. Essentially, the review chapter contains seven sections. The next section has to do with the fundamental studies; the nature of piezoelectric materials, the basic equations of piezoelectricity and the associated variational formulations are taken up. The third section reviews vibrations of piezoelectric structural elements; the works published on rods, plates, disks, shells, and

layered and composite structures are surveyed. The fourth section is devoted to the survey of works on accoustic waves and energy trapping in piezoelectric materials; the bulk waves, Rayleigh and Love waves, Stoneley and Lamb waves, and Bleustein-Gulyaev waves are considered. The fifth section deals with the studies on fracture and fatigue of piezoelectric materials, and the sixth section emphasizes the methods of numerical solutions for the equations of piezoelectric elements. In section 7, remarks on and indications of future possible trends in piezoelectricity conclude the chapter.

2- FUNDAMENTAL STUDIES

Piezoelectric synthetic materials with electric and elasto-electric nonlinearities- in particular, piezoceramics and polymers-have attracted considerable attention with regard to their nature and the origin of induced piezoelectricity in recent years [22-25]. The physical properties of some piezoceramic and polymeric materials and their dependence on certain parameters have been investigated experimentally [5,26-33] . The piezoelectric and pyroelectric behaviors of polyvinylidene fluoride, a semicrystalline polymer, have been observed after the application of high electrical stresses [27]. Measurements showed that the effect of hydrostatic pressure on the piezoelectric properties of the polymer was very small; the material was also stable with pressure cycling to a certain pressure value [28]. Lang [29] has recently compiled an extensive bibliography on piezoelectricity and pyroelectricity of polymers and their applications. The dynamic characteristics of a number of piezoceramic materials have been measured [30-31] , as has the variation of piezoelectric strain constants in ceramics under the action of uniaxial compression [32]. All the piezoelectric coefficients and elastic compliances of a crystal have been determined by the resonance method [33]. Experiments have been also done in order to investigate the electromechanical properties of piezoceramics under cyclic loading [34-36]. Another study has been conducted by use of an optical unit for the precise measurements of piezoconstants [37].

As a branch of the theory of electro-magneto-thermoelasticiy, the theory of piezoelectricity - which is an anisotropic, quasi-electrostatics, polarizable but not-magnetizable and non-conductiong field-has been well established on the basis

of the fundamental axioms of motion and those of material constitution [38]. The fundamental equations of linear piezoelectricity and thermopiezoelectricity have been recorded [39-42] and [43-46], as have those on nonlinear piezoelectricity [47-50]. In piezoelectricity, there may exist either an intrinsic nonlinearity which is peculiar to piezoelectric material or an induced nonlinearity due to the deformation of piezoelectric material. Gagnepain [51] has dealt with the elastic and piezoelectric nonlinearities in a crystal and discussed their influence on the behaviour of acoustic devices. The form invariant constitutive relations have been derived for transversely isotropic piezoelectric materials [52]. On the other hand, quasi-variational principles for the induced nonlinearity have been deduced from Hamilton's principle by the author [53-55] that generate all the three-dimensional equations of strained piezoelectric continua. He has also derived, by means of the principle of virtual work, certain variational principles, including thermal effects for a piezoelectric medium under mechanical bias [56,57]. Hailan [58] has explored the consonance of state variables of a piezoelectric body and systematically proposed the associated variational principles. Other variational principles have been formulated that may be extended to account for the equations of non-linear piezoelectricity [59] and linear thermopiezoelectricity [60]. Moreover, Kudryavtsev [61] has derived a system of linear equations for electrically polarized ceramics that differs from corresponding equations for a piezoelectric medium; it accomodates initial mechanical stresses due to the polarization.

In the presence of moving dislocations and disclinations (defects), the fundamental equations of thermopiezoelectricity have been studied for the case when the plastic deformation caused by the defects has been taken to be unrelated to the thermopiezoelectric effect of materials [62]. An electric and elastic multipole approach has been developed in studying the physical behaviour of various defects (dislocation, inhomogeneity) in finite piezoelectric media [63]. Also, the defects have been studied in an infinite medium under the influence of both mechanical and electric field loading [64]. The internal strains induced in piezoelectric crystals have been considered for given external strains produced either at constant stress or at constant electric field [65]. Electroacoustic equations have been constructed for piezoelectric powders [66] and for nonlocal piezoelectricity [67].

3- VIBRATIONS OF STRUCTURAL ELEMENTS

In investigating the vibrational characteristics of piezoelectric structural elements, the coupling of elastic field and quasi-static electric field as well as the inherent anisotropy of materials add stupendous complexities in numerical computations. Accordingly, the approximate lower order equations of elements have typically been deduced from the three-dimensional equations of piezoelectricity. The equations of elements are then approximately solved for the characteristics of any specific case. However, the vibrational characteristics have been determined by solving approximately the three-dimensional equations for a few special cases. This approach is still being developed and is not common in piezoelectricity, despite the help of large computers. Characteristics sensitive to certain parameters have been examined with the equations of elements Analytical and corroborated experimental studies have been surveyed for the vibrational characteristics of structural elements used mostly in piezoelectric devices.

RODS. Investigations concerning the analysis of piezoelectric rods have been directed toward either deriving macroscopic equations of rods [66-77] or solutions of specific problems [78-97] ; both have been studied at low-frequency vibrations. Tiersten and Ballato [68] have obtained macroscopic differential equations accounting for the nonlinear extensional motion of thin piezoelectric rods and have treated both the intermodulation and nonlinear resonance of quartz rods. Milsom and his co-workers [69,70] have described a three-dimensional mode-matching theory for coupled-mode piezoelectric rectangular bar; they showed good agreement between theory and experiments for many parameters of the bar resonator. Green and Naghdi [71] have formulated a theory of isothermal forced vibrations of piezoelectric crystal rods as a special case of their one-dimensional electromagnetic theory. Kittinger and Tichy [72] have developed a material frame theory of the influence of an electric biasing field on the extensional resonance frequency of an electroded thin piezoelectric rod. In a series of papers, the author and his co-workers [73-79] have deduced, by use of Mindlin's method of reduction [80] , various one-dimensional electroelastic equations of crystal bars from the three-dimensional equations of piezoelectricit The electroelastic equations account for all the types of extensional, flexural and torsional as well as coupled motior of bars for both low and high frequencies. The effect of mass loading of electrodes [74] , the effect of mechanical

bias [75] , the temperature effect [76] and the elastic nonlinearties [77,78] have been all taken into account, and an application to biomechanics has been described [79]. The sufficient conditions have been enumerated for the uniqueness in solutions of the linear electroelastic equations by use of either the classical energy argument [73] or the logaritmic convexity argument [77].

An analysis of the flexural-mode equations has been presented for a rod with a vibration isolator [81]. Electrode stress effects have been calculated approximately for length-extensional and flexural resonant vibrations of long, thin bars of quartz [82] . The mechanical behavior of a piezo-electric bar with an electrical voltage as well as a time-dependent flux of heat at one end has been studied [83]. A simple one-dimensional model has been used to investigate the effect of the relaxation time on the behavior of a semi-infinite piezoelectric rod under a thermal shock at its end [84]. Solov'ev [85] has recently examined the influence of the electroded zone on the natural frequency dominated thickness resonance of a piezoceramic rod of rectangular cross-section under the conditions of plain strain. The extensional vibration of a cylindrical rod with longitudinal piezoelectric coupling has been dealt with in an approximate procedure. The depolarizing-field effect has been analyzed in rods of finite and infinite lengths [86] . A detailed numerical analysis of the dispersion relations has been reported for the axisymmetric normal waves of a piezo-elecrically active bar waveguide [87] . The vibrational dissipation characteristics of a piezoceramic bar have been considered [88] , as has the electrical exication of an asymmetrically radiating bar [89].

Furthermore, Chenghao, Zheying and Yulong [90,91] have concentrated on studying the longitudinal vibrations of piezoelectric bar with lateral and longitudinal polarization and those with electric field perpendicular and paralel to the direction of vibration. The forced longitudinal vibrations of a viscoelastic piezoceramic rod with transversal polarization have been examined under the influence of harmonic electrical exication [92]. Paul and Venkatesan [93,94] have studied the vibrations of a piezoelectric solid cylinder of circular, elliptical and arbitrary cross section by use of an asymptotic method and Fourier's expansion collocation method. Other contributions are available on the dynamics of piezoelectric rods [95-99].

PLATES. Due to their extensive use as a design feature in piezoelectric devices, studies concerning the dynamic

behavior of plates are being continued to grow at a rapid pace after the publication of previous reviews [1-3,39,100-102]. A few studies were directed toward deriving the approximate, two-dimensional equations of plates on the basis of the general differential equations of piezoelectricity [103-109]. The equilibrium equations of transversely inhomogeneous piezoelectric plate have been obtained by a method of asymptotic expansion [103]; these macroscopic equations can be readily extended to account for vibrations of piezoelectric plates [104]. Similarly, the equations of low-frequency vibrations of piezocrystalline plates have been derived by the asymptotic method [105]. The governing equations and some experimental results concerning GT-type quartz crystal plates have been described [106]. By means of a variational-asymptotic method, the macroscopic equation have been established which govern the high-frequency long-wave vibrations of piezoceramic plates with thickness polarization [107]. Mindlin [108] has obtained the two-dimensional equations of motion of piezoelectric, doubly rotated, quartz plate from the three-dimensional equations of linear piezoelectricity by expansion in power series of the thickness coordinate of the plate. He then solved the macroscopic equations for forced vibrations of electroded ST-cut quartz plates and examined the effects of piezoelectric coupling and the mass of electrode coatings. By employing Mindlin's method, Lee and his co-workers [109] have also derived a hierarchical set of two-dimensional equations of motion for piezoelectric crystal plates with or without electrodes. Likewise, the author [57,110] has presented a nonlinear mathematical model for the dynamics of crystal plates with or without a mechanical bias. Moreover, the plane piezoelectric problems have been studied by expanding the static electroelastic equations into a series of trigonometric functions [111]. The stress state and the electri field distribution have been determined in a piezoelectric layer with a periodic system of electrodes at its surfaces.

Many analytical studies have been devoted to solutions of vibrations of piezoelectric plates excited in certain modes [112-123]. The thickness dominated vibrations of a plate with electrod on both its faces have been treated under both the parallel-field and perpendicular-field excitations of the plate [86]. The exact frequency equation for transverse vibrations of a piezoelectric layer has been found and then solved both numerically and by an asymptotic method [112,113]. Stevens and Tiersten [114] have calculated changes in resonant frequency with temperature for the fundamental and some of the harmonic overtone thickness

modes of an electroded contoured AT-cut quartz plate due to the thermally induced biasing stresses and strains. They showed the influences of both contouring and electrode size. The extensional and flexural biasing states have been determined by means of a variational procedure [115,116]. Sinha and Tiersten [117] have investigated thermally generated transient frequency excursions of the thickness modes of a doubly-rotated quartz plate. Further, the thickness modes of piezoelectric plates have been studied [118] using z-transform techniques [119]. These modes have been considered by Chen-hao and Zhe-ying [120] in calculating the effects of the electrode load to the resonant frequency, displacement, and stress distribution. Lee and Hou [121] have recently dealt with the computations of frequencies of thickness dominated vibrations for a doubly-rotated piezoelectric crystal strip with a pair of electrode-plated, traction-free edges. Ballato and his co-workers [122] studied all three modes of the vibrations driven by lateral fields. Stevens and Tiersten [123] have also presented an analysis of doubly rotated quartz plates vibrating in thickness modes with transverse vibration. They assumed small piezoelectric coupling and small wave numbers along the plate.

The case of pure thickness-resonance as well as that of nonlinear thickness-resonance have been studied in detail for an electroded contoured AT-and ST-cut quartz plate [124], as has the case of a vibrating polymer plate [125]. The steady-state vibrations of a thin piezoceramic plate polarized along its variable thickness have been examined [126]. The possible existence of transverse backward waves in piezoelectric plates, a relatively rare phenomenon, has also been considered [127]; critical conditions for its existence were pointed out. Furthermore, analytical works aimed at including the coupling of vibrational modes have been reported [108,128,129]. An analysis has been carried out for the coupling between the thickness-shear mode and the flexural mode of a fully electroded plate; predictions were in good agreement with experimental data [128]. Mindlin [108] has discussed the coupling of the fundamental thickness-shear mode with flexure, extension and face-shear overtones in an electroded, piezoelectric plate. In addition, Shu-chu [129] investigated the fundamental modes of coupled vibrations of piezoelectric plates and also has provided simple analytic formulas for the resonant frequencies of plates.

Various authors [130-133] have dealt with analytical investigations and experimental corroboration of vibrations of

piezoelectric plates. Ballato and his co-workers [130-131] treated crysal plates driven piezoelectrically in simple thickness modes by thickness-and lateral-directed exciting electric fields; they also reported experimental results. Suchanek [132] has examined the influence of the electrodes on the frequency of piezoelectric crystal plates by using Mindlin's theory [39]; he demonstrated both theoretically and experimentally that asymmetric electrode location rapidly reduced its elastic influence on frequency and discussed the coupling of certain modes. The frequency sensitivity to temperature has been calculated and measured for a thin quartz plate excited piezoelectrically in thickness modes [133]. Experimental results coincide well with the analytical results based on the thickness-vibration theory [100]. Bahadur and Parshad [134-136] surveyed some experimental methods for determination of mode shapes, frequencies, and amplitude of vibration of quartz crystals. The three-frequency parametric interaction of elastic waves with dispersion in a piezoelectric rectangular plate has been considered [137] ; velocity dispersion was determined from an experiment with lithium niobate crystals [137] . Additional experimental investigatinos have been carried out by Hertl et al. [138], Chenhao [139], Yushin and Beige [140], Songling and Yiyong [141], Gruzinenko et al. [142] and Bolkisev [143] . The in-plane vibration amplitudes of quartz crystals have been measured by a mechanical setup that is insensitive to environmental disturbances [138] . The piezoelectric damping configurations have been investigated for both the thickness and longitudinal vibrations of a piezoceramic plate [139] . The nonlinear electroacoustic effects have been studied in a piezoceramic slab [140].

DISKS. The ever-expanding use of disks as various active elements of piezoelectric devices has stimulated remarkable interest in vibrations of piezoceramic disks with thickness or radial polarization. Bogy and Bechtel [144] have predicted the electromechanical behavior of non-axisymmetrically loaded piezoelectric disks with electroded faces. The authors [145,146] also studied the steady vibrations of a piezoelectric disk interacting with an elastic half-space using the results of their theory [144]; the effects of the contour modes were included. Planar vibrations have been treated for a thin piezoceramic disk with metal electrodes deposited on the side surfaces of the disk and connected to an electrical load [147] . Moreover, the free radial vibrations of a piezoceramic disk polarized in thickness direction have been investigated [148] , as have its vibrational characteristics [149] , and the frequency spectrum of coupled axial and radial vibrations of finite

piezoceramic disks [150]. The stress distribution and the electric induction developed in an annular disk of inhomogeneous piezoelectric material spinning either with uniform angular velocity [151] or with time varying angular velocity [152] have been studied. Karlash [153] has dealt with energy dissipation during radial vibrations of thin circular piezoceramic disks with thickness polarization.

Using Mindlin's method [154], a system of two-dimensional equations of successively higher orders of approximation has recently been derived for vibrations of piezoelectric disks under initial stresses [155]. A system model of the thickness mode piezoelectric disk has been derived from the fundamental equations of piezoelectricity [156]. Although analytical studies with experimental justification have been pursued in this area [157-161], more work is needed. The radial modes of piezoceramic disks with open-circuit electrodes have been treated [157]. An analytical model has been proposed for evaluating the contribution of radial modes to the pulsed ultrasonic field radiated by a thick piezoelectric disk; the efficiency of the model has been shown by agreement between the results of the model and those of corresponding experiments [158]. A theoretical and experimental research has been conducted on responses to resonance and oscillation frequencies and temperature [159]. Further contributions include work on desensitization with increasing hydrostatic pressure in a flexural piezoceramic disk [160] and an empirical treatment of thickness modes in circular AT-cut quartz plates with respect to the diameter and mass loading of electrodes [161]. Ohga and his co-workers [162,163] have examined both theoretically and experimentally the flexural vibrations of a piezoelectric disk. Besides, Chongfu et al. [164] and Shouliu [165] have contributed to the radial and thickness vibrations of a piezoelectric disk, including their experimental verification.

Experimental determination of the vibrational characterictics has been reported for thin piezoceramic disk [166-175]. The mechanical resonant frequencies of disks excited electrically have been investigated by Chen [166-170]. The experimental evidence in these studies indicated that the number of purely mechanical resonances increases with decreasing disk thickness and that the domain structure affects not only the number of these resonances but also their amplitudes. Vibration velocity distributions and frequency spectra of thick disks with and without bevelling have been measured as a function of the diameter-to-thickness ratio [171]. An experimental investigation has been conducted

to determine the effect of different edge conditions on the response of piezoelectric disks [172] ; response was relatively insensitive to changes in edge conditions. The transient fields of pulsed ultrasonic sources radiating into water have been investigated using thick piezoelectric disks and broadband thickness-resonant disks as sources [173]. The spectral characteristics and amplitude distribution of the coupled flexural and thickness-shear vibrations of AT-cut quartz disks have been studied [174,175] .

SHELLS. Of the methods for reducing the three-dimensional differential equations of elastodynamics [39,104,154], the asymptotic method has been used to derive the approximate, two-dimensional equations of piezoceramic shells polarized along one of the families of coordinate lines of the middle surface [176-178]. Using again an asymtotic method, Rogacheva [179,180] has examined the free vibrations of piezoceramic shells of arbitrary shape. He has classified various types of vibrations and formulated the general theorems of electroelasticity. By the method of symbolic integration in combination with averaging of the electric potential over the shell thickness, the basic macroscopic relations have been formulated for a thin piezoelectric shell with thickness polarization and variable stiffeness [181] . These relations have then been used to examine the steady-state longitudinal vibrations of a cantilever rod of linearly varying thickness. Piezoceramic shells with thickness polarization have been treated [182] . Following the same procedure as Senik [181], the governing equations were constructed for piezoceramic gently sloping shells with meridional polarization; transverse shear deformation was considered [183] as were governing equations for piezoceramic shells with various directions of polarization [184,185]. A modified theory of piezoceramic shell polarized along the thickness has been developed by taking into account the transverse shear and the rotatory inertia [186]. By the method of series expansions in the thickness coordinate , Khoma [187,188] has derived the two-dimensional equations of piezoelectric and thermopiezoelectric shells. Similarly, the series expansions of field quantities in terms of Jacobi's polynomials have been used to construct a refined theory for axisymmetric waves in piezoceramic cylinders [189] . Green and Naghdi [190] have concerned with a theory of piezoelectric membranes as a special case of their theory of shells in which account has been taken of electromagnetic effects; this work has been mainly based on [191].

Within the limit of classical theory of elastic thin shells, that is, under the Kirchhoff-Love hypotheses of shells [191], Hongzhang [192] and Shuchu [193] have developed a theory of thin shells of radially polarized piezoceramic cylinder. As an application of Schuchu's theory, the electromechanical parameters of piezoceramic thin cylindrical tube transducers have been calculated [194] . Again, under the Kirchhoff-Love hypothesses for the mechanical variables and the corresponding hypotheses for the electrical variables, Karnaukhov and Kirichok [195] have constructed a thermomechanical theory for harmonic vibrations of viscoelastic piezoceramic shells, including the temperature dependence of materials. Chao [196,197] has presented a theory of piezoelectric and piezoceramic shells by taking a variational procedure as the basis of his derivation. Further, by a variational method of reduction [39,80,198-200], the author [1,57,201-204] has systematically derived various theories of piezoelectric shells, including the effect of mass loading of electrodes, the thermal effects and the effect of mechanical biasing stresses for both low and high frequency vibrations. He has examined the uniqueness in solutions of the governing equations of piezoelectric and thermopiezoelectric shells. On the other hand, Rogacheva [205] has dealt with the Saint-Venant type conditions in the theory of piezoelastic shells with electrodized face surfaces.

Many investigators have studied the axisymmetric and non-axisymmetric motions of piezoceramic hollow cylinders with axial, radial and circumferential polarization [206-227] through the use of special functions (e.g., [208]), power series representation of field variables in the radial coordinate [209], numerical integration of the initial equations by the method of discrete orthogonalization [210], the finite element method and alike. The axisymmetric motion of radially polarized piezoelectric hollow cylinders has been investigated [206,211,212]. The longitudinal [212-215] and circumferential [216] as well as torsional wave motions [217] of a piezoelectric solid cylinder have been studied in detail. Loza and his co-workers [218-220] have dealt with the propagation of axisymmetric and non-axisymmetric waves in a piezoceramic hollow cylinder with radial and axial polarizations, and he [200] has also treated the axisymmetric acoustoelectric wave propagation in the cylinder with circumferential polarization. The dynamic stress state has been determined in a compound piezoceramic hollow cylinder with thickness polarization [221]. Paul and Venkatesan [222] have considered the longitudinal and flexural modes of a hollow circular cylinder of piezoelectric ceramics. The forced axisymmetric vibrations

of a cylindrical piezoceramic shell with radial polarization [223] and the nonstationary vibrations of the shell with circular polarization [224] have been studied. Burdess [225] has presented the equations of motion for a thin piezoelectric cylinder gyroscope. Then he has determined the dynamic response of gyroscope to constant and harmonic rates of turn. The natural free oscillations of a class of cylindrical piezoelectric ceramics and the correspondin displacement amplitudes have been obtained [226] . Tingrong [227] has reported a new measurement method and used it for measuring the material constants of a radially polarized thin piezoceramic cylindrical tube.

The interaction effects of the radiation load and various modes of vibrations of a piezoceramic cylindrical shell have been examined for the case when the shell with thickness polarization is partially in contact with an acoustic medium and surrounded by a soft shield [228] . In a similar case, Babaev and Savin [229] have examined the action of transient electrical signal on the motion of a thin-walled cylindrical piezoceraimic shell with circumferential polarization, which is surrounded by and filled with a compressible fluid. Shu-chu [193] has dealt with the scattering of plane waves by a radially polarized piezoceramic cylinder using Green's function and the method of separation of variables. Loza and Shul'ga [230,231] have analyzed the axisymmetric vibrations of a hollow piezoceramic sphere with radial polarization. The dissipative heating of a viscoelastic piezoceramic ball with temperature-dependent properties has been investigated [232]. The radially polarized ball performs radial vibrations in an acoustic medium under harmonic excitation. Additional works have included some analytical and experimental results for fezoceramic spherical and cylindrical shells [233-237].

LAYERED AND COMPOSITE STRUCTURAL ELEMENTS. With their desirable vibration characteristics for ultrasonic applications, piezoelectric layered and composite structural elements have been widely used in different technologies. The use of composite piezoelectric materials and the basic ideas underlying their sum and product properties have been described [238-241] . Basically two types of macromechanical models exist for the analysis of these structural elements: the effective modulus model and the effective stiffeness model, as in composites [242]. The effective modulus model replaces an element by a representative homogeneous medium with the aid of the averaged material constants of element constituents. This model, although it is relatively simple, omits the coupling of adherent layers in laminated composites.

The extension of the Lagrange and Kármán models of plates as well as the Kirchhoff-Love models of shells to crystal lamina elements leads to their effective modulus model. Along this line, a macromechanical model of regular piecewise-homogeneous structures with piezoceramic matrices has been presented [243]. The effective constants of randomly inhomogeneous piezoactive (piezoelectric and piezomagnetic) ceramics have been determined [244]. Similarly, the effective properties of composite piezoelectric ceramics stochastically reinforced by spheroidal inclusions have been considered; from this follow as limit cases materials with laminated, unidirectional fibrous and granular structure [245]. On the other hand, the effective stiffeness model combines both the physical and geometrical properties of lamina constituents and incorporates all their essential features. Within the concept of this model, the one-dimensional and two-dimensional approximate equations of laminae have been consistently derived, including a theorem of uniqueness [1,74]. As an extension of classical models, the macroscopic relations of electroelasticity have been derived for multilayer piezoceramic plates and shells [246-255], their steady-state vibrations have been reported in some special cases.

Notably, Parton and Senik [246] have derived macroscopic equations of multilayer piezoceramic shells with thickness polarization of the layers. They have also treated the vibrations of a shallow spherical shell of three layers. Likewise, Karnaukhov and his coleagues [247-250] have constructed the governing equations of laminated piezoceramic plates and shells by taking into account the geometrical nonlinearity, the effect of temperature, and, in particular, the effect of viscosity. The viscosity effect is important for polymeric materials with polarization in hydroacoustics and, in fact, piezoceramic materials are viscoelastic in terms of their mechanical nature [251]. The electroviscoelastic layered shells have been polarized through their thickness or in one coordinate direction. The effect of temperature has been also considered in describing the dynamic behavior of multilayered piezoceramic shells with thickness polarization under harmonic excitation [252]. Loza and his co-workers [253] have described an algorithm in investigating the propagation of nonaxisymmetric acoustoelectric waves in a layered circular cylinder with axial, radial or circumferential directions. Shu-chu [255] has treated the coupled vibrations of a composite cylinder in a way convenient to engineering design and estimation. In addition, the radial mode oscillations have been analyzed for a piezoelectric element consisting of

several concentric cylinders [256] . The influence of height of a hollow, two-layer piezoceramic cylinder has been investigated on the spectrum of its resonance frequencies [257] . Moreover, the anti- and axi-symmetric elektromechanical wave propagations have been considered in long bone [258] and [259] where the bone has been modelled as a two-layered cylindrical shell.

Other studies have involved a close examination of resonances and modelling of composite piezoelectric plates [260-269]. Auld and his co-workers [260,261] developed a Floquet theory of wave propagation in periodic composites that has been shown to agree with experiment. The thickness-extensional vibrations of a composite plate [262] have been studied in detail by use of a variational principle due to Tiersten [39] . The flexural vibrations of a piezoceramic laminae have been numerically investigated [263] . Ting-rong [264-266] has dealt with the forced vibrations of piezoceramic composite circular plate excited either with voltage or with homogeneous pressure. The effect of attachment conditions has also been considered on the parameters of a two-layered piezoceramic plate [267] . The geometry of composite plates has been analyzed by the finite difference method [268] . The stress-strain state of layered-stepped piezoelectric disk has been analyzed under flexural [270] and coupled flexural-shear oscillations [271]. Also, a method of iteration has been presented for the coupled dynamic analysis of a layered circular disk with thickness polarization. The influence of the dependence of material properties on temperature has been considered [272] . On the other hand, a general transfer matrix description of arbitrarily layered piezoelectric structures with two electrodes has been derived [273] . Besides, research has been conducted in the area of composite and layered piezoelectric rods [272,274,275] . All the elastic, piezoelectric and dielectric constants have been analytically derived for a one- and two-dimensional multilayered structures and some experiments have been carried out [274] . Also, the results of an experimental study of vibrations of composite piezoelectric rod with longitudinal polarization have been reported [275] .

4- WAVES IN CRYSTALS

In piezoelectric crystals, the interaction between the elastic waves and the electromagnetic waves is weak because

their velocities are very different. Therefore, the two types of wave propagation have been always treated independently in linear piezoelectricity [276]. Attention has recently been paid to the interaction of electromagnetic and acoustic waves due to the nonlinear piezoelectric effect [277] and that in piezoelectric plates [278]. Gilinskii and Vdovin [279] have described the propagation of accusto-electromagnetic pulses in a bounded piezoelectric crystal, including the coupling of elastic and electromagnetic waves. However, only the propagation of elastic waves is surveyed herein. Some reviews and treatises have been mentioned [2,3,8,12-18,276,280-289]. Work done on bulk waves in unbounded medium and that on surface waves in semi-infinite medium is reviewed in this section. Bleustein-Gulyaev, Rayleigh and Love, and Stoneley and Lamb waves are also discussed, as is energy trapping.

BULK WAVES. The research on bulk acoustic waves and especially on surface skimming bulk waves and reflected bulk waves has been carried out in microwave acoustic devices [290-295]. Josse and Lee [290] have reported an analytical solution that describes the analysis of excitation, interaction and detection of bulk and surface waves on piezoelectric crystals. He and his co-workers [291-293] have theoretically analyzed the reflection of bulk acoustic waves, the amplification of surface skimming SH waves and the amplification and convolution of reflected bulk acoustic waves in rotated Y-cut quartz. The excitation and detection of surface-generated bulk waves have been treated [294,295]. The synchronous interactions of bulk acoustic waves have been investigated in piezoelectric insulator crystals with spatially inhomogeneous structure [296]. The bulk-surface electroacoustic waves have been considered at the interface of a piezoelectric with a semiconductor [297]. The conversion of bulk strain waves has been examined at a frequency boundary in a semi-infinite piezoelectric crystalline medium [298]. The reflection of bulk acoustic waves has been studied in a layered piezoelectric (insulator)-gap-semiconductor structure, as has experimentally the linear and nonlinear acoustoelectronic interaction in such a structure [299,300]. An interactive computer-aided analysis has been described for calculating the main properties of bulk acoustic waves in materials of arbitrary anisotropy and piezoelectricity [301] as well as the sensitivity of bulk waves to the temperature effect [302]. The numerical calculations of the anisotropy of electric-field control of the velocity of bulk acoustic waves have been reported in piezoelectrics

having a sillenite structure [303]. The properties of bulk and surface acoustic waves have been considered in piezoelectric crystals with both intrinsic and induced nonlinearities. The nonlinear propagation of a finite amplitude wave and the propagation of a small amplitude wave have been treated in a strained piezoelectric crystal [304]. By choice of crystal cut and wave propagation direction, bulk waves may propagate nearly parallel to the crystal surface; these waves have been termed as shallow bulk acoustic waves or as surface skimming bulk waves. Research progress and prospects can be found in a notable article [305]. Theoretical results have been reported for certain piezoelectric crystals; propagation characteristics are given on the reflection of surface skimming bulk waves [306] as are experimental results for nonlinear interactions when bulk acoustic waves reflect off the boundary of a piezocrystal in a layered structure . Other analytical and experimental contributions have been reported [308-313].

SURFACE ACOUSTIC WAVES. Surface sound (acoustic) waves in solids have wide applications in piezoindustry; hence, they have been thoroughly investigated both theoretically and experimentally [7-19,276,280-283]. An analysis of excitation of surface waves with piezoelectric layers has been presented [314,315]. The relations between the energy flux, group and phase velocities of surface acoustic waves in an arbitrary semi-infinite piezoelectric medium have been established for various types of boundary conditions; they have also been established for Stoneley waves in piezoelectric bicrystals [316]. The energy fluxes along the boundary in the reflection of a transverse plane wave have been examined [317]. The dispersion curves of straight-crested wave propagating in a ST-cut quartz plate have been obtained by use of the equations of piezoelectric crystal plates due to Lee et al. [109]; the agreement has been very close between the theoretical prediction and the experimental results [318]. The scattering of acoustic waves by transverse and longitudinal modes has been elucidated in a piezoelectric half-space [319]. In addition, the scattering of surface waves has been consistently dealt with, as has the interaction between surface waves in piezoelectric media and electrode structures [320]. The carrier drift has been shown to exert a significant influence on the scattering of a transverse wave by a cylindrical cavity in a hexagonal piezoelectric [321]. Also, an approximate method of analysis [322] and a variational analysis [323] have been introduced in studying the scattering properties of surface acoustic waves. A quantitative determination of

diffraction effects has been made in surface accoustic wave harmonic generation [324]. Moreover, the reflection of a transverse wave from the surface of a hexagonal piezoelectric crystal has been considered and the effect of piezomoduli variations on the reflection phase shift has been examined [325]. The specific characteristics of the reflection of a transverse wave have been discussed at a piezoelectric-semiconductor interface under acoustic bonding conditions [326]. A variational analysis of the reflection of surface waves by arrays of reflecting grooves has been presented [327].

Viktorov and Pyatakov [328] have dealt with the main specific features of surface acoustic waves on cylindrical surfaces of piezoelectric crystals, including the influence of surface curvature, crystal anisotropy, piezoelectric effect and conductivity of cylinder material. Detailed computational results have been reported for the viscous attenuation and velocity of surface acoustic waves propagating along various directions in selected orientations of quartz [329]. An analysis of thermal effects has been carried out for the propagation characteristics of surface acoustic waves [330-332]. On the other hand, the interaction between surface electrodes and piezoelectric crystals, a topic of importance for various surface wave devices, has been investigated [333-342]. Longitudinal and transverse acoustoelectric effects have been discussed in a layered semiconductor-piezoelectric structure [335]. In a series of papers, V'yun [336-340] has dealt with the acoustoelectric interaction of surface acoustic waves in layered piezoelectric-semiconductor structures. He has developed an impedance method in studying the acoustoelectric interaction with weak electromechanical coupling [336], considered the intrinsic nonlinear interaction of surface waves [339] and reported the characteristic properties of the hysteresis of acoustoelectric interaction [340]. In addition, a variational approach has been used to analyze the parameters that describe the interaction of surface acoustic waves with short-circuited metal strip gratings [341]. A coupled amplitude equation has been developed that has been applied to interactions arising from the weak nonlinearities of materials supporting surface acoustic waves [342]. Alippi [343] has studied qualitatively the effects associated with nonlinear acoustic propagation in piezoelectric crystals with special reference to the case of surface acoustic waves. He performed experiments on the effects. A treatment of second harmonic generation of surface waves in piezoelectric

solids [344] has been presented by use of the nonlinear electroelastic equations [345]. Analytical expressions have been derived for the velocity and attenuation of surface acoustic waves in layered structure [346], and a compact formulation of these waves has been given by extending the surface Green function matching analysis [347]. The generalized Green's function has been used in the analysis of surface waves [313]. A numerical method of computation has been described for acoustic wave generation [348] and acoustic wave properties [349].

In the presence of induced nonlinearity in piezoelectric media, the velocity of surface acoustic waves is dependent upon the nature of biasing stresses and strains and mode of wave propagation. The nonlinear properties of surface acoustic waves have been discussed; in particular, the harmonic generation and the amplitude shift have been examined as a function of propagation direction [304]. The temperature and stress induced effects on the propagation characteristics of surface elastic waves have been investigated [302,350-356], as has the influence of intrinsic stresses [353,354]. Sinha et al. [355,356] have described some analytical and experimental results on the stress and temperature induced effects on the surface wave propagation in crystalline quartz. The propagation of surface acoustic waves has been experimentally studied in ion-implanted lithium niobate [357-359], as has the influence of a biasing electric field on the propagation [360,361].

BLEUSTEIN-GULYAEV WAVES. This type of surface waves has no counterpart in a purely elastic material; it is a face-shear type of elastic waves that arise at the free surface of a piezoelectric crystal. The dispersion relation of Bleustein-Gulyaev waves has been investigated along symmetry directions of surfaces and interfaces, either metalized or non-metalized, of piezoelectric cubic crystals. It has been shown that no Bleustein-Gulyaev waves can exist along certain direction of a surface [362]. The propagation characteristics of waves have been studied in a piezoelectric crystal [363] as well as its nonlinear constitutive relations [364]. Kudryavtsev and Parton [365] have dealt with the excitation of Bleustein-Gulyaev shear surface acoustic waves by two ribbon electrodes of finite length and determined the characteristics of these waves. The effect of reflection and transmission of a Bleustein-Gulyaev wave has been studied theoretically [366]. This effect has been also investigated experimentally [367]. The surface and bulk waves with emphasis on a Bleustein-Gulyaev wave have been considered in elastic semiconductors

in a bias electric field [368,369]. The scattering of waves has been examined analytically in normal incidence on an ideally conducting strip on the surface of a hexagonal crystal [370]. The scattering at the edge of a metal film, gaps of various width, a wide electrode, and the rectangular end of an acoustic line have been studied [371] Additional contributions on waves have been made by Lyumibov [372], who dealt with the conditions of existence and dispersion of elastic surface waves due to the piezoelectric effect in a free, infinite crystal plate. The generation of a Bleustein-Gulyaev wave has been treated in oblique incidence of a shear bulk wave on the nonhomogeneous boundary of a piezoelectric half-space [373]. Leaky or pseudo-surface Bleustein-Gulyaev and Bleustein-Gulyaev waves have also been described in detail [374,375].

RAYLEIGH AND LOVE WAVES. Rayleigh wave is a mode of acoustic wave propagation that may exist at the free surface of an elastic half-space, while Love wave propagates between the interface of a thin layer and an elastic half-space. Chenghao and Dongpei [376] have recently dealt with the scattering of Rayleigh wave through a groove on the surface of a piezoelectric crystal; they also analyzed the scattered field by the boundary perturbation method. Approximate dispersion relations for Rayleigh and Love waves have been obtained in an elastic half-space with a thin piezoelectric film [377]. The generation of the second harmonic of a Rayleigh wave has been investigated in a layered structure [378]. A theoretical analysis of shear horizontal surface Love waves has been performed on rotated Y-cut quartz crystal |374| and on an isotropic substrate with a piezoelectric layer [379]. The dispersion equation has been derived and analyzed for surface Love waves propagating in a semi-infinite piezoelectric substrate on which an isotropic solid dielectric layer has been deposited [380]. The propagation characteristics of Love waves in a periodically-layered structure have been investigated [381]; the growth rates of waves depend on nonlinearly on the number of periods in the structure. The influence of such parameters as a biasing electric field and a temperature increment has been considered on the propagation of transverse Love surface waves [382]. Morocha [383] has studied the propagation of pure transverse waves along an interface between two piezoelectric media; he also dealt with the propagation of gap waves in an asymmetrical parallel-plate waveguide.

STONELEY AND LAMB WAVES. Stoneley waves propagate at a plane interface between two perfectly bonded, elastic half-space and Lamb waves propagate in thin layers. The fundamental characteristics of a Stoneley surface acoustic wave generated by an electrode transducer have been calculated at the interface of a piezoelectric and nonconducting liquid [384]. The effect of piezoelectric moduli and that of electrical boundary conditions have been investigated on the existence, velocity and kinematic properties of Stoneley waves at the interface of two piezoelectric media [385]. Adler [386] has delat with the electromechanical coupling to Lamb modes in piezoelectric plates. The propagation of Lamb waves has been examined in a planar layer made of piezolectrics of hexagonal syngony [387].

ENERGY TRAPPING. Due to its excellent features, the concept of energy trapping has been increasingly utilized in the design of piezoelectric devices. Milsom and his colleagues [388] have developed a three-dimensional mode-matching theory of piezoelectric plated bars, including both the mass loading and electrical shorting effects of the electrodes; the results were in good agreement with experiments. They also found that energy trapping varies with the cross-sectional aspect ratio of the bar. An analysis has been made of a piezoelectric plate driven into thickness-extensional trapped energy vibrations by the application of a voltage to strip electrodes and radiating into an adjacent fluid [389]. All previous treatments ignored radiation into the surrounding fluid. Tiersten and his co-workers have considered various aspects of energy trapping [389-394]. They dealt with the transient thermally-induced frequency excursions at AT-cut and SC-cut quartz crystal [390], analyzed thickness-extensional trapped energy modes in a thin piezoelectric film on silicon structure [391], and studied the change in orientation of a zero-temperature contoured SC-cut quartz crystal with the radius of the contour [392]. A simple theoretical model of trapped energy resonators with circular electrodes that utilize monoclinic crystal plates has recently been proposed for thickness-wave solutions in the vicinity of cutoff frequencies [395], as has a model with rectangular electrodes for analyzing the effets of tab electrodes on an AT-cut plate [396]. A simple method has been provided for predicting frequencies of energy trapped modes of thickness vibrations in piezoelectric rectangular and circular plates [397]. Peach [398] has determined the design characteristics of AT-cut and SC-cut quartz crystal trapped energy resonators by a variational method. Recently, Détaint et al. [399]

have addressed to the energy trapping in plane and corrugated resonators and reported the experimental and computed results.

5- FRACTURE AND FATIGUE

Anaytical investigations concerning the strength and failure of piezoelectric materials are of recent origin, and, in fact, began with the discovery and manufacturing of piezoceramics. Among investigations on the use of methods of electroelasticity, Parton and Kudryavtsev [400] have described the fracture of thermopiezoelectric materials, have studied the crack growth of arbitrary form, and have determined the condition of crack propagation in certain cases. A method has been proposed for determining the conjugate mechanical and electrical fields in a piezoelectric medium weakened by a curvilinear tunnel cut, including a numerical example [401]. The intensity factors of electrical and mechanical quantities have been calculated for the longitudinal shear of a piezoelectric medium with a tunnel notch [402]. Kuz'menko, Pisarenko and Chushko [403] have predicted the fatigue life (endurance) of piezoceramic elements from the characteristics of a subcritical crack growth; the lower bound of endurance given by the prediction agrees well with the measured data. Development of microcracks has been considered in a piezoceramic half-plane with two boundary electrodes [404]. Further, Parton [405] has contributed on the subject as an extension of his previous work [406]. Purely experimental studies have been directed toward the determination of the fracture toughness [407,408] and fatigue failure [409] of piezoceramics.

6- METHODS OF NUMERICAL SOLUTIONS

Among the methods of numerical analysis in continuum physics, the finite element and boundary element methods have long been used for solutions of elastodynamic problems. The literature in this area has grown enormously since the evolution of digital computers. However, only in the last few years, the finite element method began to be used to solve dynamic problems of piezoelectric crystals. Allik and Hughes [410] and Oden and Kelley [411] have described

the finite element method as a universal numerical method of piezoelectric analysis. Using the finite element method Naillon and his coleagues [412,413] have thoroughly described an analysis of piezoelectric structures, including some applications. Kovalev and his colleagues [414] have introduced an approximate method of numerical solutions for problems of electroelasticity on the basis of variational-difference methods, of which the finite element method is a modification. The finite element method has also been used to calculate the linear and nonlinear propagation modes in a piezoelectric surface wave guide [415,416]. A numerical approach based on the finite element method has been described for the analysis of periodic waveguides for acoustic waves and, in particular, of propagation characteristics of SH surface waves and Rayleigh waves [417]. The vibrations of piezoelectric bar have been simulated by use of a finite element method [418]. Xiaoqi and Quichang [419] have developed a finite element-equivalent circuit method to compute the vibration and acoustic radiation of a piezoelectric composite rod. The dynamic influence on the flexible cantilever beam with distributed active piezoelectric damper has been considered by the finite element method [420]. Besides, a staircase model has been presented for the analysis of a tapered piezoelectric bar [421]; the theoretical and experimental results have been reported. Again, using the finite element method, the vibrational mode spectrum in an axisymmetric piezoelectric disk has been characterized [422,423]. Karnaukhov and Kozlov [424] have described the method for an investigation of the electromechanical behavior of thermo-electro-viscoelastic solids of revolution under harmonic loading. They have also performed numerical calculations for a piezoceramic viscoelastic cylinder with radial polarization. Moreover, the addition of piezoelectric properties to structural finite element programs has been achieved by matrix manipulations [425,426]. The finite element method has been reviewed for electroelastic vibration and static analyses of piezoelectric structural elements [427].

The method of Laplace transforms, the method of z-transforms and the method of fast Fourier transforms [428-431] have been applied to solutions of dynamic problems of piezoelectric crystals. By the method of Laplace transforms, Zhang and his colleagues [432] have obtained the complete analytic solutions of the transient behavior of a transmitting thickness-mode piezoelectric infinite plate. They gave the physical interpretations of complete solutions as well. By the method of z-transforms, rapid solutions have

been proposed to the transient response of piezoelectric elements [433]. By the method of fast Fourier transforms, the transient response of a piezoelectric cylinder [193] and sphere [434] has been treated. Polak et al. [435] have discussed mathematical and computational aspects of device modelling that may be applied to the analysis of piezoelectric elements.

The boundary element method has been described for solutions of piezoelectric problems [436], although specific problems remain to be solved. The finite element method has been applied to electric and magnetic field problems, including a number of applications [437]. A brief account of recent algorithms has been given for electromagnetic computation in two and three dimensions and at low frequencies [438]; this and the finite element method can be readily extended for solutions of some dynamic problems of piezoelectric crystals.

7. CONCLUSIONS

The aim of this paper is to review the open literature related to the dynamic applications of piezoelectric crystals since 1983. Representative work, both theoretical and experimental, has been surveyed that deal with vibrations of rods, plates, disks, shells and laminae; with bulk waves, surface acoustic waves, energy trapping, fatigue and fracture; and with methods of numerical solutions. This review should be of value to readers seeking guidance; it also provides a challenge to interdisciplinary researchers in the field of piezoelectricity.

As is apparent from the representative literature cited, a considerable amount of valuable works has been done on waves and vibrations in piezoelectric crystals. However, most of works has been devoted to analytical solutions of specific problems using conventional numerical methods; little of this analytical work has experimental corroboration ; very little work relates only to experiments and basic research. Analytical and experimental works, including applications, that deserve special attention have to do with polar and nonlocal piezoelectric materials and piezoelectric powders and alike [439]. Efforts are needed to develop a relativistic and stochastic approach to dynamic problems of as well as to the thermodynamics and stability of

piezoelectric crystals [440-442]. Investigations are anticipated to address more challenging problems of inelastic and nonlinear behavior, fracture, reliability and endurance of piezoceramics. Moreover, due to their computational efficiency, the finite element method has to be applied extensively to dynamic problems of piezoelectric crystals, as has the boundary element method even though no specific applications are yet available. Lastly, there still exists a need for experimental works to determine some constitutional behavior and sensivity of piezoelectric materials and to corroborate theoretical results. In wiew of its current trend in technology, opportunities appear to be plentiful and potentially fruitful for future work on the subject.

This is an extended version of the recent survey paper [443] with an updated bibliography.

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CHAPTER 2

CERTAIN INTEGRAL AND DIFFERENTIAL TYPES OF VARIATIONAL PRINCIPLES IN NONLINEAR PIEZOELECTRICITY

ABSTRACT

Various forms of variational principles are developed so as to generate, as Euler-Lagrange equations, the fundamental differential equations of nonlinear piezoelectricity. To begin with, Hamilton's principle is rigorously applied to the motion of an electroelastic solid with small piezoelectric coupling, and an associated variational principle is readily derived. This two-field variational principle yields some of the fundamental equations of the piezoelectric solid, and it contains the remaining fundamental equations as its constraints. Then, by use of the dislocation potentials and Lagrange undetermined multipliers (Friedrichs's transformation), the variational principle is augmented for the motion of piezoelectric solid region with an internal surface of discontinuity. Likewise, to incorporate the constraints into the two-field variational principle, Friedrichs's transformation is again applied, and hence a unified variational principle is shown to produce the fundamental equations of electroelastic solid with small piezoelectric coupling. Further, similar variational principles are formulated for the incremental motion of piezoelectric solid that is initially under stress.

1- INTRODUCTION

In Describing the physical behavior of piezoelectric solid media, the elastic field is taken to be dynamic and the electric field to be static, and both the fields are considered to be linear with respect to electromagnetic propagation phenomena. This linear quasi-static approximation provides an extremely accurate description of the propagation of small-amplitude waves in, and the small vibrations of, the solid media. However, the linear approximation becomes inadequate in high amplitudes, and hence should be taken into account the intrinsic nonlinearity and/or the induced nonlinearity due to the peculiarity and the deformation of solid media, respectively. In fact, the nonlinear phenomena were already demonstrated experimentally and investigated analytically for some dynamic problems of piezoelectric solids and especially quartz crystals ([1],

[2], and references therein). In quartz, which is probably the most widely used crystal, the electromechanical coupling, that is, the piezoelectric effect is weak, and hence, the electrical behavior is taken to be linear while all the elastic nonlinearities are included. On the other hand, the presence of initial stresses may significantly affect the dynamic characteristics of quartz crystals, and accordingly the linear approximation should be modified [3]. In this study, various forms of variational principles are derived for the motion of an electroelastic solid with small piezoelectric coupling (e.g., quartz) and that of a piezoelectric solid subjected to initial stresses.

The governing equations for the motion of piezoelectric solid are constructed on the basis of the general principles of electroelasticity. They consist of the divergence (field) equations, the constitutive relations, the gradient equations and the appropriate boundary and initial conditions. Of these fundamental equations, the field equations are originally stated in global form through the integral expressions of mechanical and electrical balance laws. The global field equations are essential and general due to their physical nature, and their local (differential) counterparts can be stated under some regularity and local differentiability conditions. The constitutive relations appropriately express the peculiarities of piezoelectric solid, and they are, in general, stated in differential form under certain rules and invariant requirements. However, these relations should be stated in integral (global) form for the case of a nonlocal piezoelectric solid in which the nature of long-range intermolecular forces is taken into account. The rest of the fundamental equations are always given in differential form.

Besides their global and local forms, the fundamental equations of piezoelectric solid can be alternatively expressed in variational form by means of the stationarity of appropriate functionals. In stating the fundamental equations, all the three forms are, of course, equivalent, and interdependent, and they can be deduced from one another. From the standpoint of computation, the global form is inappropriate, the differential form is widely used in analyzing the motion of piezoelectric solid, and the variational form has certain advantages over the others. In the latter form, the fundamental equations are generated as the Euler-Lagrange equations of variational principles which may be contrived in certain cases. In order to derive a variational principle, a general principle of physics (e.g., Hamilton's principle and the principle of virtual work) is often taken as a

starting point in lieu of experienced quesswork. Of variational principles, an integral variational principle (e.g., Hamilton's principle) admits an explicit functional, whereas a differential variational principle (e.g., D'Alembert principle) denies it. Even Hamilton's principle becomes a differential variational principle for the case when the nonconservative forces do exist. On the other hand, the principle of virtual work and the like, by definition, cannot have explicit functionals due to their postulated statements in terms of infinitesimals called virtual displacements and virtual work. The differential variational principles are especially valuable from the standpoint of succintly summarizing the fundamental equations, deducing lower order field equations and obtaining approximate direct solutions. In addition to these features, the integral variational principles are useful in finding bounds formulae and in studying existence and convergence proofs of solutions. In closing, the differential variational principles can be contrived almost in all cases, whereas the integral variational principles are generally applicable to the linear and self-adjoint fundamental equations, and their existence can be tested by use of Fréchet derivatives [4],[5].

In deriving variational principles, Hamilton's principle [6], [7], which was originally derived for a discrete mechanical system and later extended by Kirchhoff [8] to a continuum, was used successfully in dynamics, solid and fluid mechanics, and piezoelectricity. The application of this principle to a continuous medium always leads to a variational principle that generates only the field equations and the associated natural boundary conditions, and hence it implements the remaining fundamental equations of a medium as its constraints. The constraint(subsidiary) conditions make difficult a free and simple choice of approximating (trial or coordinate) functions in computation, and accordingly, in many instances, it is desirable to remove them. There exists a number of ways for the inclusion of constraint conditions into the variational principle, and a simple way of implementing is to use Friedrichs's transformation [9]-[11]. Other noteworthy ways to be used for the removal of constraints are the adjoint equation method or the method of the mirror equation advocated by Morse and Feschbach [12] in continuum physics, the quasi-variational method of Biot [13] in thermodynamics, the restricted variational method or the method of local potential put forward by Rosen [14], and Glansdorff and Prigogine [15], and the method of convolution due to Gurtin [16] in elasticity. Among those, Friedrichs's transformation is applicable to holonomic as

well as nonholonomic conditions as shown by Lanczos [11], and it is particularly valuable and of wide use in removing constraints in both elasticity and electroelasticity (e.g., [17]-[19]). In fact, due to its versatility and clarity in application, Friedrichs's transformation is also used herein in modifying Hamilton's principle into the unified variational principles of nonlinear piezoelectricity.

In piezoelectricity, Tiersten and Mindlin [20], Tiersten [21], [22], EerNisse [23],[24] and Holland and EerNisse [25]-[27] primarily developed certain variational principles that were elaborated in [22], [28]. Starting with Hamilton's principle, Tiersten [21] derived a two-field variational principle, and then he modified it through Lagrange undetermined multipliers in order to obtain an extended variational principle. This variational principle yields, as its Euler-Lagrange equations, the field equations and the associated boundary conditions as well as the pertinent jump conditions for a piezoelectric bounded region containing an internal surface of discontinuity. Also, Vekovishcheva [29] established, by experienced guesswork, a few variational principles in the theory of electroelasticity, as did the author [30],[31]. Especially, the initial and jump conditions were excluded in [29],[31], and these conditions were taken into account by the author [32] who was guided by the work [18]. The variational principles in [32] generate all the fundamental equations of piezoelectricity, analogous to the variational principles of Hu-Washizu and *Hellinger-Prange*-Reissner [33] in elasticity. Further, in the light of Gurtin's method of convolution [34], another variational principle with no constraints was formulated by Sandhu and Pister [35].

To include thermal effect, Mindlin [36] discussed a variational principle in thermopiezoelectricity by extending Biot's [37] variational principle for the thermoelastic case. Nowacki [38] and recently Chandrasekharaiah [39] formulated some variational principles with constraints through Hamilton's principle. The unconstrained variational principles were derived by the author [40]-[42] who followed both the methodology described in [8] and [34]. Moreover, Kudriavtsev, Parton, and Rakitin [43] established a condition that was the generalization of the fracture variational principle in piezoelectric solids, as did Parton [44] and the author [45]. Lastly, mention should be made of the variational principles for a piezoelectric solid under initial stresses [46],[47] and those for an electroelastic solid with small piezoelectric coupling [48],[49]. These variational principles were obtained by use of either Hamilton's

principle or the principle of virtual work together with Friedrichs's transformation. As regards the relevant literature on variational principles in piezoelectricity, the reader may be referred to the list of references in [2], [40, 41].

In what follows, the fundamental equations are recorded for a nonlinear electroelastic solid with small piezoelectric coupling in the next section. Hamilton's principle is stated for the nonlinear electroelastic solid, and then by performing suitable variations and integrations by parts, a two-field variational principle is derived that yields the field equations and the associated natural boundary conditions, in Section 3 . By use of the dislocation potentials and Lagrange undetermined multipliers, Hamilton's principle is modified in Section 4 , and hence an extended variational principle is established for the electroelastic region with an internal surface of discontinuity. The two-field variational principle of Section 3 is similarly augmented through Friedrichs's transformation and a unified variational principle is obtained in Section 5. This variational principle is shown to generate the fundamental equations of nonlinear electroelastic solids. Moreover, by comparing the principles derived, some variational principles are presented for a piezoelectric solid subjected to initial stresses, in Section 6 . The last section is devoted to special cases, concluding remarks and further needs of research.

Note that throughout the paper, the usual indicial notation is freely used in a three-dimensional (3-D) Euclidean space E. In this space, the x_i- system is identified with a fixed, right-handed system of Cartesian convected (intrinsic) coordinates. Einstein's summation convention is implied over all repeated Latin indices, unless they are put within parentheses. A comma stands for partial differentiation with respect to the indicated space coordinate and a superposed dot for time differentiation. Also, an asterisk denotes prescribed quantities, and a boldface bracket indicates the jump of an enclosed quantity across a surface of discontinuity. The symbol B(t) refers to a regular, finite and bounded region B contained in E at time t, $\bar{B}$ denotes the closure of the region B with its boundary surface ∂B, and $\bar{B} \times T$ represents the Cartesian product of the region B and the time integral $T=[t_o, t_1)$. As regards new quantities, they are essentially defined whenever they first appear.

2 - SUMMARY OF NONLINEAR PIEZOELECTRIC EQUATIONS

In this section, the fundamental nonlinear equations of an electroelastic solid with small piezoelectric coupling are stated in differential form. In accordance with the small piezoelectric coupling, the nonlinear elastic terms are included only, and the electric and electroelastic terms are kept linear. The fundamental equations were derived in [1], [50], [51], and they are expressed herein for completeness and convenience.

In the space E, let $B+\partial B$ stand for an arbitrary, simply-connected, finite and bounded region of the electroelastic solid [52], and also let the region B be referred to by a fixed, right-handed system of Cartesian convected coordinates x_i. The entire boundary surface ∂B of B consists of the complementary regular subsurfaces $(\partial B_t, \partial B_u)$ or $(\partial B_q, \partial B_\phi)$, and the unit outward vector normal to ∂B is denoted by n_i. Further, let $\bar{B}XT$ represent the domain of definitions for the functions (x_i, t).

Now, the 3-D fundamental equations are recorded as the following divergence equations:

$$t_{ij,i} = \rho a_j \quad \text{in } \bar{B}XT \tag{1a}$$

$$t_{ij} = \tau_{ij} + T_{ij} = \tau_{ik}(\delta_{jk} + u_{j,k}) \tag{1b}$$

$$e_{ijk}\tau_{jk} = 0 \quad \text{in } \bar{B}XT \tag{1c}$$

$$D_{i,i} = 0 \quad \text{in } \bar{B}XT \tag{2}$$

with the definitions

t_{ij}	asymmetric Lagrangian stress tensor
τ_{ij}	symmetric Kirchhoff stress tensor
$T_{ij} = \tau_{ik} u_{j,k}$	symmetric Maxwell electrostatic stress tensor
ρ	density of the undeformed body
a_i	Lagrangian acceleration vector $(=\ddot{u}_i)$
u_i	mechanical displacement vector
δ_{ij}	Kronecker delta
D_i	electric displacement vector
e_{ijk}	permutation symbol.

Equation (1) stands for the nonlinear stress equations of motion and (2) for the linear charge equation of electrostatics.

Gradient equations:

$$S_{ij}=1/2(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}) \text{ in } \bar{B}XT \tag{3a}$$

$$S_{ij}=e_{ij}+1/2(e_{ki}+w_{ki})(e_{kj}-w_{kj}) \tag{3b}$$

$$e_{ij}=1/2(u_{i,j}+u_{j,i}),w_{ij}=1/2(u_{i,j}-u_{j,i}) \tag{3c}$$

and

$$E_i=-\phi,_i \quad \text{in } \bar{B}XT \tag{4}$$

with the definitions

S_{ij}	Lagrangian strain tensor
e_{ij}	linear strain tensor
w_{ij}	rotation tensor
ϕ	electric potential
E_i	quasi-static electric field vector

Equation (3) represents the nonlinear strain-mechanical displacement relations and (4) the linear electric field-electric potential relations.

Constitutive equations:

$$T_{ij}=1/2\left(\frac{\partial H}{\partial S_{ij}}+\frac{\partial H}{\partial S_{ji}}\right) \quad \text{in } \bar{B}XT \tag{5}$$

$$D_i=\frac{\partial H}{\partial E_i} \quad \text{in } \bar{B}XT \tag{6}$$

where $H=U-E_iD_i$ is electric enthalpy; and U is potential energy density. A quartic form of the electric enthalpy is recorded in the form.

$$H=1/2C_{ijkl}S_{ij}S_{kl}-1/2C_{ij}E_iE_j-C_{ijk}E_kS_{ij}$$
$$+\ 1/6C_{ijklmn}S_{ij}S_{kl}S_{mn}$$
$$+\ 1/24C_{ijklmnrs}S_{ij}S_{kl}S_{mn}S_{rs} \qquad (7)$$

where C_{ijkl}, C_{ijklmn}, $C_{ijklmnrs}$ are second-order, third-order, and fourth-order elastic constants; respectively, C_{ijk} is piezoelectric strain constants; and C_{ij} is dielectric permittivity.

Of these constants, the elastic constants refer to free constants since they describe the strain-stress relations when the electric field is absent, while the remaining constants refer to clamped constants [53]. Further, the usual symmetry relations hold for the material constants, namely,

$$C_{ijkl}=C_{jikl}=C_{klij},\ C_{ijk}=C_{ikj},\ C_{ij}=C_{ji}$$
$$C_{ijklmn}=C_{ijmnkl}=C_{klijmn}=C_{jiklmn} \qquad (8)$$
$$C_{ijklmnrs}=C_{klijmnrs}=C_{mnijklrs}=C_{rsijklmn}=C_{jiklmnrs}$$

In view of (5)-(7), the constitutive equations for the symmetric stress tensor and the electric displacement vector are expressed in respective forms:

$$\tau_{ij}=C_{ijk}S_{kl}-C_{ijk}E_k+1/2\ C_{ijklmn}S_{kl}S_{mn}$$
$$+\ 1/6\ C_{ijklmnrs}S_{kl}S_{mn}S_{rs}$$
$$D_i=C_{ijk}S_{jk}+C_{ij}E_j,\quad \text{in } \bar{B}XT \qquad (9)$$

Boundary conditions:

$$t_i^*-n_jt_{ji}=t_i^*-n_j\tau_{jk}(\delta_{ik}+u_{i,k})=0 \quad \text{on } \partial B_tXT \qquad (10)$$

$$u_i-u_i^*=0 \quad \text{on } \partial B_uXT \qquad (11)$$

$$\sigma^*-n_iD_i=0 \quad \text{on } \partial B_\sigma XT \qquad (12)$$

$$\phi-\phi^*=0 \quad \text{on } \partial B_\phi XT \qquad (13)$$

where $t_i=n_j t_{ji}$ is the stress vector; n_i is the unit outward vector normal to ∂B; and $\sigma_i=n_i D_i$ is the surface charge; where the stresses are taken to be prescribed on ∂B_t, the mechanical displacements on ∂B_u, the surface charge on ∂B_σ and the electric potential on ∂B_ϕ.

Initial conditions:

$$u_i(x_j,t_0)-v_i^*(x_j)=0 \qquad \text{in } B(t_0) \qquad (14)$$
$$u_i(x_j,t_0)-w_i^*(x_j)=0$$

and

$$\phi(x_i,t_0)-\psi^*(x_i)=0 \qquad \text{in } B(t_0) \qquad (15)$$

Jump conditions:

$$\nu_i[\![t_{ij}]\!]=\nu_i[\![\tau_{ik}(\delta_{jk}+u_{j,k})]\!]+t_j^{\alpha}=0 \quad \text{on } SXT \quad [\![u_i]\!]=0 \qquad (16)$$

and

$$\nu_i[\![D_i]\!]=Q,\ [\![\phi]\!]=0 \quad \text{on } SXT \qquad (17)$$

where t_i^{α} is the applied prescribed surface traction; Q is the electric surface charge density; S is the material surface of discontinuity. In these equations, the surface traction and charge density with zero intensity are considered, that is, $t_i^{\alpha}=Q=0$, the conventional notation $[\![\chi_i]\!]$ for $(\chi_i^+-\chi_i^-)$ is introduced, and also ν_i denotes the unit normal vector directed from the (-) to the (+) side of S.

Equations (1)-(15) completely describe the nonlinear behavior of nonlocal, nonpolar, and nonrelativistic elecroelastic solid with small piezoelectric coupling, and (16) and (17) arise at a material surface of discontinuity. The existence conditions have yet to be established in solutions of the initial mixed boundary value problems defined by the fundamental equations (1)-(15). Nevertheless, the boundary and initial conditions (10)-(15) were shown to be sufficient for the uniqueness in solutions of the fully linearized problems of fundamental equations [22].

3- THE APPLICATION OF HAMILTON'S PRINCIPLE TO THE ELECTROELASTIC SOLID

In piezoelectricity, Hamilton's principle was rigorously expressed, and then the associated linear variational principles were readily deduced from it in [22]. The treatment was restricted only to the nonrelativistic case, and also the polar as well as nonlocal effects were excluded. Likewise, this principle is now applied and an associated variational principle is derived for an electroelastic solid with small piezoelectric coupling.

Hamilton's principle states that the action integral is stationary between two instants of time t_o and t_1, namely

$$\delta\mathcal{L}= \delta\int_T L dt+ \int_T \delta W dt=0 \tag{18}$$

where L is the Lagrangian function and δW is the virtual work done by external mechanical and electrical forces for all the admissible motions of electroelastic solid, that is, the motions which are compatible with their given constraint conditions and of which the end points are taken to be coterminus with those of the actual motion. In (18), the Lagrangian function is defined by

$$L =\int_B \left[K-H(S_{ij},E_i)\right] dV \tag{19}$$

for the regular region of electroelastic solid $B+\partial B$ with its entire boundary surface ∂B, and the kinetic energy density K is expressed by

$$K= 1/2\ \rho\dot{u}_i\dot{u}_i \tag{20}$$

and the electric enthalpy by (5)-(7). The virtual work per unit area done by the prescribed surface tractions t_i^* and surface charge σ^* is given by

$$\delta W=\int_{\partial B} (t_i^*\delta u_i+\sigma^*\delta\phi) dS \tag{21}$$

After inserting (19)-(21) into (18), one arrives at the variational equation of the form

$$\delta\mathcal{L}= \int_T \{ \delta\int_B \left[1/2\rho\dot{u}_i\dot{u}_i-H(S_{ij},E_i)\right] dV + \int_{\partial B}(t_i^*\delta u_i+\sigma^*\delta\phi) dS \} dt=0 \tag{22}$$

where all variations vanish at $t=t_0$ and $t=t_1$. Then, by carrying out the indicated variations, utilizing the fact that the operation of variation commutes with that of differentiation and integrating by parts with respect to time, this equation takes the form

$$\partial\mathcal{L}= \int_T dt \int_B(-\rho a_i \delta u_i - \tau_{ij}\delta S_{ij} + D_i \delta E_i)dV$$

$$+\int_T dt \int_{\partial B}(t_i^* \delta u_i + \sigma^* \delta\phi)dS=0 \tag{23}$$

Here, the variations are assumed to obey the axiom of conservation of mass, namely,

$$\delta(\rho dV)=0 \tag{24}$$

and the constraint conditions on them, the constitutive equations (5) and (6) are employed, and the symmetry of Kirchhoff stress tensor is considered. Now, by substituting the nonlinear strain-mechanical displacement relations (3) and the linear electric field-electric potential relations (4) into (23) and applying the divergence theorem for the regular region $B+\partial B$, and then combining terms in the surface and volume integrals, one finally obtains a two-field variational principle as

$$\partial\mathcal{L}_1\{\Lambda_1\}= \int_T dt \int_B(L_{1j}\delta u_j + L_2\delta\phi)dV$$

$$+\int_T dt \int_{\partial B}(L_{1j}^* \delta u_j + L_2^* \delta\phi)=0 \tag{25a}$$

with the admissible state $\Lambda_1=\{u_i,\phi\}$ and the definitions

$$L_{1j}=|\tau_{ik}(\delta_{jk}+u_{j,k})|,_i - \rho a_j \tag{25b}$$

$$L_2 = D_{i,i} \tag{25c}$$

$$L_{1j}^* = t_j^* - n_i\tau_{ik}(\delta_{jk}+u_{j,k}) \tag{25d}$$

$$L_2^* = (\sigma^* - n_i D_i). \tag{25e}$$

In deriving this variational principle, the condition

$$\partial u_i = \delta\phi = 0 \tag{26}$$

in $B(t_o)$ and $B(t_1)$ is imposed. In the variational principle (25), since the variations δu_i and $\delta\phi$ of the admissible state Λ_1 are arbitrary and independent inside the volume B and on the boundary surface ∂B, one has the nonlinear stress equations of motion (1), the linear charge equation of

electrostatics (2) and the natural boundary conditions (10)-(13). Conversely, if these equations are met, the variational principle (25) is evidently satisfied. The admissible state Λ_1 of (25) is subjected to the remaining fundamental equations of electroelastic solid and the condition (26) as its constraints.

The constrained variational principle (26) can be used in solving approximately the boundary-value problems defined by the fundamental equations of nonlinear electroelastic solid, provided that the initial conditions (14) and (15) may be left out of account by a variety of numerical techniques (e.g., [54],[55]). However, any approximating solution must satisfy the constraints of (25); this feature of Hamilton's principle was discussed very thoroughly by Tiersten [18]. Further, it is of interest to note that the two-field variational principle (25) can be extracted from the principle of virtual work [49], and it contains some of earlier variational principles of [21], [22], [25], [30]-[32],[47] as special cases whe the nonlinear terms are discarded.

4 - A VARIATIONAL PRINCIPLE FOR DISCONTINUOUS ELECTROELASTIC FIELDS

Now, paralleling to the derivation of the two-field variational principle in the previous section, a variational principle is deduced from Hamilton's principle for a region B of nonlinear electroelastic solid, intersected by a fixed surface of discontinuity S. This internal surface of discontinuity splits the bounded and finite region $B+\partial B$ with its entire boundary surface ∂B, and each region is labeled by $\alpha\,(\alpha=1,2)$. The region B_α is bounded by the boundary surface ∂B_α and S, and hence, $\partial B_1 \cup \partial B_2 = \partial B$ and $\partial B_1 \cap \partial B_2 = 0$. Let the mechanical displacements and the electric potential undergo a jump across the discontinuity surface S. Then applying the global laws of balance postulated in electroelastodynamics and using the generalized Green-Gauss theorem fc a field χ_i, namely,

$$\int_{B-S} \chi_{i,i}\, dV = \int_{\partial B-S} \nu_i \chi_i\, dS - \int_S \nu_i (\chi_i^{(2)} - \chi_i^{(1)})\, dS \qquad (27)$$

one obtains the local balance of laws (1) and (2) and also the jump conditions (16) and (17). Here, the exponent α is used to designate the region $B_\alpha + \partial B_\alpha$. On the other hand,

a surface of discontinuity $S(t)$ with a velocity field is of special importance in piezoelectric moving media, and the jump conditions should be accordingly modified when there exists such a moving surface.

To begin with, let L_α and δW_α denote respectively the generalized Lagrangian function and the virtual work for the region $B_\alpha+\partial B_\alpha$. Thus Hamilton's principle (18) is rewritten in the form

$$\delta\mathcal{L}=\sum_{\alpha=1}^{2}\left[\delta\int_T L_\alpha dt+\int_T \delta W_\alpha dt\right]=0 \tag{28}$$

for the nonlinear electroelastic region $B+\partial B$ with a fixed surface S. As before, carrying out the indicated variations, it can be readily shown that this equation leads to a variational principle in the form

$$\begin{aligned}\delta\mathcal{L}(\Lambda_\alpha)=\int_T dt\int_B \sum_{\alpha=1}^{2}(L_{1j}^{(\alpha)}\,\delta u_j^{(\alpha)}+L_2^{(\alpha)}\,\delta\phi^{(\alpha)})\,dV\\ +\int_T dt\int_{\partial B}\sum_{\alpha=1}^{2}(L_{1j}^{*(\alpha)}\,\delta u_j^{(\alpha)}\\ +L_2^{*(\alpha)}\,\delta\phi^{(\alpha)})\,dS=0\end{aligned} \tag{29}$$

in terms of the quantities defined in (25) for each region. This principle yields the divergence equations and the associated natural boundary conditions of each region as its Euler-Lagrange equations. To incorporate the jump conditions into (29), the variational principle is modified slightly by Friedrichs's transformation [9]-[11], and hence a dislocation potential Δ, each constraint as a zero times a Lagrange multiplier, is added in the Lagrangian, namely,

$$\begin{aligned}\delta\mathcal{L}=\delta\int_T dt\sum_{\alpha=1}^{2}L_{(\alpha)}\\ +\int_T dt\sum_{\alpha=1}^{2}\delta W_{(\alpha)}+\delta\int_T \Delta dt=0\end{aligned} \tag{30}$$

with

$$\Delta=\int_S(\lambda_i[\![u_i]\!]+\lambda[\![\phi]\!])\,dS \tag{31}$$

where λ_i and λ are the Lagrange multipliers to be determined. In performing the indicated variations in (30), all the variations of field variables and those of Lagrange multipliers are unconstrained everywhere except at $t=t_0$ and t_1 where the usual condition (26) of Hamilton's principle is imposed. Now, carrying out the variations in a way anologous to those in (22) and applying the generalized Green-Gauss theorem (27)

which is valid only up to and not across the surface of discontinuity S, (30) takes the form

$$\delta\mathcal{L}_2=\delta\mathcal{L}\{\Lambda_\alpha\}$$

$$+\int_T dt\int_S\{[\lambda_j+\nu_i\tau_{ik}^{(2)}(\delta_{jk}+u_{j,k}^{(2)})]\delta u_j^{(2)}$$

$$-[\lambda_j+\nu_i\tau_{ik}^{(1)}(\delta_{jk}+u_{j,k}^{(1)})]\delta u_j^{(1)}$$

$$+(\lambda+\nu_i D_i^{(2)})\delta\phi^{(2)}$$

$$-(\lambda+\nu_i D_i^{(1)})\delta\phi^{(1)}+\delta\lambda_i(u_i^{(2)}-u_i^{(1)})$$

$$+\delta\lambda(\phi^{(2)}-\phi^{(1)})\}dS=0 \tag{32}$$

Here, the volumetric and surface variations $\delta u_i^{(\alpha)}$ and $\delta\phi^{(\alpha)}$ are free in the region B_α and on the surfaces ∂B_α and S, and the variations $\delta\lambda_i$ and $\delta\lambda$ are arbitrary on the discontinuity surface S. Thus, for (32) to be satisfied in all these admissible variations, the integrand of each variation must vanish and this gives the divergence equations (1) and (2) and the natural boundary conditions (10) and (12) for each region $B_\alpha+\partial B_\alpha$, namely

$$L_{1j}^{(\alpha)}=L_2^{(\alpha)}=0,\ L_{1j}^{*(\alpha)}=L_2^{*(\alpha)}=0;\ \alpha=1,2 \tag{33}$$

and the jump conditions (16) and (17) at their interface, and also the conditions of the form

$$\lambda_j+\nu_i\tau_{ik}^{(\alpha)}(\delta_{jk}+u_{j,k}^{(\alpha)})=0 \tag{34}$$

$$\lambda+\nu_i D_i^{(\alpha)}=0 \tag{35}$$

By solving (34) and (35) for λ_i and λ, the Lagrange multipliers are identified as tractions and surface charge in their most appropriate form

$$\lambda_j=-1/2\ \nu_i\sum_{\alpha=1}^{2}\tau_{ik}^{(\alpha)}(\delta_{jk}+u_{j,k}^{(\alpha)}),$$

$$\lambda=-1/2\ \nu_i\sum_{\alpha=1}^{2}D_i^{(\alpha)} \tag{36}$$

Then, the substitution of (36) into (30) results in an augmented variational principle as follows:

$$\delta\mathcal{L}_2\{\Lambda_2\}=\int_T dt \sum_{\alpha=1}^{2}\int_{B_\alpha}(L_{1j}^{(\alpha)}\delta u_j^{(\alpha)} + L_2^{(\alpha)}\delta\phi^{(\alpha)})dV$$
$$+\int_T dt \sum_{\alpha=1}^{2}\int_{\partial B_\alpha}(L_{ij}^{*(\alpha)}\delta u_j^{(\alpha)}$$
$$+ L_2^{*(\alpha)}\delta\phi^{(\alpha)})dS$$
$$+ \int_T dt\int_S 1/2\nu_i\{[\![\tau_{ik}(\delta_{jk}+u_{j,k})]\!]$$
$$\cdot (\delta u_j^{(1)}+\delta u_j^{(2)})$$
$$-[\delta\tau_{ik}^{(1)}(\delta_{jk}+u_{j,k}^{(1)})$$
$$+\delta\tau_{ik}^{(2)}(\delta_{jk}+u_{j,k}^{(2)})] [\![u_i]\!]$$
$$+ (\delta\phi^{(1)}+\delta\phi^{(2)})[\![D_i]\!]$$
$$- (\delta D_i^{(1)}+\delta D_i^{(2)})[\![\phi]\!]\}dS=0 \qquad (37a)$$

In a compact form, it is

$$\delta\mathcal{L}_2\{\Lambda_\alpha\} =\int_T dt \sum_{\alpha=1}^{2}[\delta\int_{B_\alpha}(1/2\rho^{(\alpha)}\dot{u}_i^{(\alpha)}\ \dot{u}_i^{(\alpha)}$$
$$-H^{(\alpha)})dV$$
$$+\int_{\partial B_\alpha}(t_i^{*(\alpha)}\delta u_i^{(\alpha)}+\sigma^{*(\alpha)}\delta\phi^{(\alpha)})dS]$$
$$-\delta\int_T dt\int_S 1/2\nu_i\{(t_{ij}^{(1)}+t_{ij}^{(2)}) [\![u_j]\!]$$
$$+ (D_i^{(1)}+D_i^{(2)})[\![\phi]\!]\} dS=0 \qquad (37b)$$

with the admissible state

$$\Lambda_2= \{u_i^{(\alpha)},\tau_{ij}^{(\alpha)};\ t_{ij}^{(\alpha)},\phi^{(\alpha)},D_i^{(\alpha)}\} \qquad (37c)$$

which leads to the divergence equations and the natural boundary conditions for each region and the jump conditions as its Euler-Lagrange equations. The differential type of variational principle (37a) and in particular its form (37b) is very useful for approximation, and also it covers some of earlier variational principles for the linear case (e.g., [22],[41] and references therein). In deriving (37), the inclusion of the jump conditions through Friedrichs's transformation is a classical example of implementing subsidiary conditions in a variational principle[10],[11]. However, there is a slight difference between this and the

classical one, that is, the constraint conditions (16) and (17) are boundary constraints and λ_i and λ are functions on the discontinuity surface, while the constraint conditions treated in the classical way are either domain or isoperimetric constraints. Moreover, to introduce a modified Lagrangian (30) in lieu of the original Lagrangian (28) is also physically plausible, that is to say, the modification of the Lagrangian by the dislocation potentials is not merely a matter of mathematical method but has a very real physical significance. The modification of the Lagrangian on account of Friedrichs's transformation represents the Lagrangian that is responsible for the maintenance of the given constraint conditions.

5 - VARIATIONAL PRINCIPLES FOR NONLINEAR PIEZOELECTRICITY

In the previous sections, three variational principles (25), (29), and (37) of nonlinear piezoelectricity are deduced from Hamilton's principle, and they impose certain constraint conditions upon their admissible states. In general, neither a priori satisfaction of such conditions nor by introducing additional unknowns in terms of Lagrange multipliers is desirable in computation. Thus, it is preferable to use variational principles with as few constraints as possible, as suggested by computational economy. Of the constraints of variational principles, the jump conditions (16) and (17) are already relaxed in (37) with the help of Friedrichs's transformation. Now, the constraint conditions of (25) are similarly removed, and then a unified variational principle is derived for the motion of an electroelastic solid with small piezoelectric coupling.

To incorporate its constraint conditions (3)-(6), (11), and (13) into the two-field variational principle (25), the dislocation potentials are introduced as follows:

$$\Delta_{11}=\int_B \lambda_{ij}\left[S_{ij}-1/2(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})\right]dV \tag{38a}$$

$$\Delta_{22}=\int_B \mu_i(E_i+\phi_{,i})\,dV \tag{38b}$$

$$\Delta_{33}=\int_{\partial B_u} \lambda_i(u_i-u_i^{\star})\,dS+\int_{\partial B_\phi}\mu(\phi-\phi^{\star})\,dS \tag{38c}$$

where λ_{ij}, μ_i, λ_i, and μ are the Lagrange multipliers to be determined. The dislocation potentials (38) are added to (22) with the result

$$\delta\mathcal{L}_3 = \int_T dt \int_B (1/2\rho\dot{u}_i\dot{u}_i - H)\,dV$$

$$+\int_T dt \int_{\partial B} (t_i^{\star}\delta u_i + \sigma^{\star}\delta\phi)\,dS$$

$$+\int_T \delta\Delta_{ii}\, dt = 0 \tag{39}$$

Then, as in (22), all the indicated variations in this equation are treated, and a variational equation is obtained in the form

$$\delta\mathcal{L}_3 = \int_T dt \int_B \Big[-\rho a_i \delta u_i - 1/2\Big(\frac{\partial H}{\partial S_{ij}} + \frac{\partial H}{\partial S_{ji}}\Big)\delta S_{ij}$$

$$- \frac{\partial H}{\partial E_i}\,\delta E_i\Big]dV$$

$$+\int_T dt \int_{\partial B} (t_i^{\star}\delta u_i + \sigma^{\star}\delta\phi)\,dS$$

$$+\int_T dt \int_B \Big[\delta\mu_i(E_i + \phi_{,i})$$

$$+\mu_i(\delta E_i + \delta\phi_{,i})\Big]dV$$

$$+\int_T dt \int_B \{\delta\lambda_{ij}\big[S_{ij} - 1/2(u_{i,j} + u_{j,i}$$

$$+ u_{k,i}u_{k,j})\big]$$

$$+ \lambda_{ij}\big[\delta S_{ij} - (\delta u_{i,j} + u_{k,i}\delta u_{k,j})\big]\}\, dV$$

$$+\int_T dt \int_{\partial B_u} \big[\delta\lambda_i(u_i - u_i^{\star}) + \lambda_i\delta u_i\big]dS$$

$$+\int_T dt \int_{\partial B_\phi} \big[\delta\mu(\phi - \phi^{\star}) + \mu\delta\phi\big]dS = 0 \tag{40}$$

By exactly the same way as in (23), the Green-Gauss theorem is applied, the conditions (24) and (26) are imposed, and then the terms that belong to a certain variation are collected. Thus, one finally obtains the fundamental equations of electroelastic solid and the Lagrange multipliers identified by

$$\lambda_{ij} = \tau_{ij},\ \mu_i = -D_i,\ \lambda_i = t_i = n_j t_{ji},\ \mu = \sigma = n_i D_i \tag{41}$$

since the volumetric and surface variations of Lagrange multipliers and field variables are free in the region B and on the surface portions of its boundary surface ∂B. By substituting (41) into (40), the unified variational principle is found in the form

$$\begin{aligned}\delta \mathcal{L}_3\{\Lambda_3\} = \int_T dt \int_B (&L_{1j}\delta u_j + L_2\delta\phi + L_{ij}\delta S_{ij} \\ &+ L_i\delta E_i + K_{ij}\delta\tau_{ij} + K_i\delta D_i)\,dV \\ &+ \int_T dt \int_{\partial B_t} L^{*}_{1j}\delta u_j dS \\ &+ \int_T dt \int_{\partial B_u} L^{*}_{i}\delta t_i dS \\ &+ \int_T dt \int_{\partial B_\sigma} L^{*}_{2}\delta\phi dS \\ &+ \int_T dt \int_{\partial B_\phi} K^{*}\delta\sigma dS = 0\end{aligned} \tag{42}$$

with the admissible state $\Lambda_3 = \{u_i, S_{ij}, \tau_{ij}, t_i; \phi, \sigma, E_i, D_i\}$ and the definitions (25b)-(25e) and the denotations as

$$L_{ij} = \tau_{ij} - 1/2\ \left(\frac{\partial H}{\partial S_{ij}} + \frac{\partial H}{\partial S_{ji}}\right) \tag{43a}$$

$$L_i = D_i + \frac{\partial H}{\partial E_i} \tag{43b}$$

$$K_{ij} = S_{ij} - 1/2(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \tag{43c}$$

$$K_i = -(E_i + \phi_{,i}) \tag{43d}$$

$$L^{*}_{i} = (u_i - u^{*}_{i}) \tag{43e}$$

$$K^{*} = (\phi - \phi^{*}) \tag{43f}$$

The variational principle (42) yields the fundamental equations of nonlinear electroelastic solid, namely,

$$L_{1j} = L_2 = L_{ij} = K_{ij} = K_i = L_i = 0 \quad \text{in} \quad \bar{B}XT \tag{44a}$$

$$\begin{aligned}&L^{*}_{1j} = 0 \quad \text{on } \partial B_t XT,\ L^{*}_{i} = 0 \quad \text{on } \partial B_u XT, \\ &L^{*}_{2} = 0 \quad \text{on } \partial B_\sigma XT,\ K^{*}_{i} = 0 \quad \text{on } \partial B_\phi XT\end{aligned} \tag{44b}$$

as its Euler-Lagrange equations, and it is stated below.

Variational principle: Let $B+\partial B$ denote a regular, finite and bounded region of the space E, with its piecewise smooth boundary surface $\partial B (=\partial B_u \cup \partial B_t = \partial B_\sigma \cup \partial B_\phi$ and $\partial B_u \cap \partial B_t = \partial B_\sigma \cup \partial B_\phi = 0)$ and its closure $\bar{B}(=B \cup \partial B)$. Then, of all the admissible states $\Lambda = \{u_i, S_{ij}, \tau_{ij}, t_i; \phi, \sigma, E_i, D_i\}$, which satisfy the initial conditions (14) and (15) and the condition (26) as well as the symmetry of stress tensor τ_{ij} and the usual continuity and differentiability conditions of field variables, if and only if, that admissible state Λ that satisfies the nonlinear stress equations of motion (1), the linear charge equation of electrostatics (2), the nonlinear strain-mechanical displacement relations (3), the electric field-electric potential relations (4), the nonlinear constitutive equations (5) and (6), and the natural boundary conditions (10)-(13), is determined by the variational principle

$$\delta \mathcal{L}_3\{\Lambda_3\}=0 \tag{45}$$

of (42) as its appropriate Euler-Lagrange equations.

In view of the virtual work defined in (21), the variational equation (45) represents a differential type of variational principle in nonlinear piezoelectricity. However, the variational principle (45) can be readily written in a compact form by

$$\begin{aligned}
&\delta \mathcal{L}_3\{u_i, S_{ij}, \tau_{ij}, t_i; \phi, E_i, \sigma, D_i\} \\
&= \delta\{\int_T dt \int_B [1/2\rho \dot{u}_i \dot{u}_i - H(S_{ij}, E_i)]\, dV \\
&+ \int_B \tau_{ij}[S_{ij} - 1/2(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j})]\, dV \\
&- \int_B D_i(E_i + \phi_{,i})\, dV \\
&+ \int_{\partial B_u} t_i(u_i - u_i^*)\, dS + \int_{\partial B_t} t_i^* u_i\, dS \\
&+ \int_{\partial B_\phi} \sigma(\phi - \phi^*)\, dS \\
&+ \int_{\partial B_\sigma} \sigma^* \phi\, dS\} = 0
\end{aligned} \tag{46}$$

Here, the functional $\mathcal{L}_3$ does exist and an integral type of variational principle is indeed supplied as an alternative description of the fundamental equations of nonlinear piezoelectricity, and it has, of course, all the fruitful features of classical variational principles (e.g. [11] and, in particular, [56], [57] where a lucid discussion of the search of variational principle was given).

On the other hand, the Legendre transformation $\mathcal{H}(\tau_{ij}, D_i)$ of the electric enthalpy $H(S_{ij}, E_i)$, that is, the complementary electric enthalpy may be introduced as

$$\mathcal{H}(\tau_{ij}, D_i) = \tau_{ij} S_{ij} - E_i D_i - H(S_{ij}, E_i) \tag{47}$$

for the case when the Hessian of H does not vanish. Then, inserting (47) into (45) and imposing the relations (3) and (4), one arrives at another variational principle in the form

$$\delta \mathcal{L}_4\{u_i, \tau_{ij}, t_i; \phi, \sigma, D_i\} = 0 \tag{48a}$$

with its Lagrangian

$$\begin{aligned}
\mathcal{L}_4 &= \int_T dt \int_B \left[1/2 \rho \dot{u}_i \dot{u}_i + \mathcal{H}(\tau_{ij}, D_i)\right] dV \\
&- \int_T dt \int_B 1/2 \tau_{ij} (u_{i,j} + u_{j,i} + u_{k,i} + u_{k,j}) dV \\
&- \int_T dt \int_B D_i \phi_{,i} dV + \int_T dt \int_{\partial B_u} t_i (u_i - \overset{\star}{u}_i) dS \\
&+ \int_T dt \int_{\partial B_t} \overset{\star}{t}_i u_i \, dS \\
&+ \int_T dt \int_{\partial B_\phi} \sigma (\phi - \overset{\star}{\phi}) dS \\
&+ \int_T dt \int_{\partial B_\sigma} \overset{\star}{\sigma} \phi dS
\end{aligned} \tag{48b}$$

This integral type of variational principle leads to the nonlinear stress equations of motion (1), the linear charge equation of electrostatics (2), the constitutive equations in the inverted form

$$S_{ij} = 1/2 \left(\frac{\partial \mathcal{H}}{\partial \tau_{ij}} + \frac{\partial \mathcal{H}}{\partial \tau_{ij}}\right), \quad E_i = -\frac{\partial \mathcal{H}}{\partial D_i} \quad \text{in } \bar{B} \mathrm{XT} \tag{49}$$

and the boundary conditions (10)-(13), as its appropriate Euler-Lagrange equations.

Moreover, replacing the electric enthalp H by its quartic version (7) in the variational principle (45), one obtains a slightly different variational principle with the new definitions L_{ij} and L_i of (43a) and (43b) in the form

$$L_{ij} = \tau_{ij} - (C_{ijkl}S_{kl} - C_{ijk}E_k + 1/2C_{ijklmn}S_{kl}S_{mn}$$

$$+ 1/6\ C_{ijklmnrs}S_{kl}S_{mn}S_{rs})$$

$$L_i = D_i - (C_{ijk}S_{jk} + C_{ij}E_j) \qquad (50)$$

which leads to the constitutive equations (9).

In the unified variational principle (45), by use of the fundamental lemma of the calculus of variations, one obtains all the fundamental equations of nonlinear piezoelectricity, that is, (1)-(6) and (10)-(13), but (1c) and the initial conditions (14) and (15). Conversely, if the fundamental equations except (1c), (14) and (15) are met, the variational principle is evidently satisfied. The variational principle can be further extended, following the method indicated by Tiersten [18], so as to adjoin the initial conditions as well as the jump conditions (16) and (17) into (45); the result is a differential type of variational principle [47], [49].

In closing, the unified variational principle (45) does agree with and contains, as special cases, certain earlier variational principles operating on some of the field variables (e.g., [21]-[27], [30]-[32], [47] and references cited therein). The variational principle can be specialized to derive several new variational principles in nonlinear piezoelectricity. Of these principles, the variational principle (25) operating only on the electric potential and the mechanical displacements is recorded as

$$\delta \mathcal{L}_1(u_i, \phi) = 0 \qquad (51)$$

and its reciprocal variational principle is expressed by

$$\delta \mathcal{L}_5(S_{ij}, \tau_{ij}, t_i; E_i, \sigma, D_i)$$

$$= \int_T dt \int_B (L_{ij}\delta S_{ij} + L_i \delta E_i + K_{ij}\delta\tau_{ij} + K_i \delta D_i)\, dV$$

$$+ \int_T dt \int_{\partial B_u} L_i^{*} \delta t_i\, dS + \int_T dt \int_{\partial B} K^{*} \delta\sigma\, dS = 0 \qquad (52)$$

in the notation of (43). The reciprocal variational principle (52) is simply obtained by discarding the term involving δu_i and $\delta\phi$ in (45). The Euler-Lagrange equations of this principle represent the divergence equations (3) and (4), the constitutive equations (5) and (6), and the boundary conditions (11) and (13), and the principle has the admissibility conditions (1),(2),(10),(12),(14), and (15) as its constraints. Whereas the two-field variational principle (51) has (1),(2), (10), and (12) as its Euler-Lagrange equations, and its admissibility conditions are (3),(6),(11), and (13)-(15). Thus, the variational principle (52) is the reciprocal of the variational principle (51), since the roles of admissibility conditions and the Euler-Lagrange equations are interchanged.

6 - VARIATIONAL PRINCIPLES FOR INCREMENTAL MOTIONS IN PIEZOELECTRICITY

Initial stress or initial strain is a new design feature, and their introduction may be effectively utilized to control the performance of some piezoelectric devices. Their presence may significantly change the dynamic behavior of piezoelectric elements as revealed by many investigations (e.g., [46], [58], and in particular, [59]). Nevertheless, no efforts have been made to formulate the governing equations of a piezoelectric medium under initial stress through variational principles. Thus, this section is addressed to the derivation of variational principles for the strained piezoelectric medium. In what follows, Hamilton's principle is used in deriving a two-field variational principle which generates only the divergence equations and the associated natural boundary conditions. Then, the variational principle is augmented by applying Friedrichs's transformation so as to incorporate the remaining fundamental equations of medium. Before proceeding further, the three-dimensional fundamental equations of strained medium are recorded for ease of reference as follows.

In the space E, consider a regular, finite, and bounded region $B_o+\partial B_o$ of piezoelectric medium, with its boundary surface ∂B_o, in its initial state. The initial state is taken to be self-equilibrating following a static loading in the natural (or stress-free) state of region at time t=0. The fundamental equations of initial state can be expressed by the same equations (1)-(15) when the dynamic terms are dropped out and the quantities of this state are designated

by a zero index (e.g., [60]). Then, by an elastic motion superimposed upon the state $B_o+\partial B_o$, which is subjected to initial stress, the piezoelectric region acquires its spatial state $B+\partial B$ at time $t=t_o$. For this type of incremental motions, (23) deduced from Hamilton's principle can be written in the form

$$\delta \mathcal{L}_6 = \delta\int_T dt \int_B K dV$$
$$- \int_T dt \int_B \left[(\tau^0_{ij} + \tau_{ij})\delta S_{ij} - D_i \delta E_i \right] dV$$
$$+ \int_T dt \int_{\partial B} \left[(t^{0*}_i + t^*_i)\delta u_i + \sigma^* \delta\phi \right] dS = 0 \qquad (53)$$

where the kinetic energy is defined by

$$K = 1/2\, \rho \dot{u}_i \dot{u}_i \qquad (54)$$

the Lagrangian strain tensor S_{ij} and the electric field vector E_i by

$$S_i = e_{ij} + 1/2 u_{k,i} u_{k,j} \quad \text{in } \bar{B}XT \qquad (55)$$

$$E_i = -\phi_{,i} \quad \text{in } \bar{B}XT \qquad (56)$$

the surface tractions and surface charge by

$$t^0_j = n_i t^0_{ij},\ t_j = n_i (\tau_{ij} + \tau^0_{ik} u_{j,k}) \qquad (57)$$

$$\sigma = n_i D_i \qquad (58)$$

and the constitutive relations by

$$\tau_{ij} = C_{ijkl} S_{kl} - C_{ijk} E_k \quad \text{in } \bar{B}XT \qquad (59)$$

$$D_i = C_{ijk} S_{jk} + C_{ij} E_j \quad \text{in } \bar{B}XT \qquad (60)$$

In the above equations, u_i is the incremental displacement vector, τ^0_{ij} is the symmetric initial stress tensor, τ_{ij} is the symmetric incremental stress tensor, and C_{ijkl}, C_{ijk}, and C_{ij} are the material constants of piezoelectric medium. By implying the usual arguments on the increments of field variables [33] and taking into account the stress equations of motion and the associated boundary conditions in the initial state, and then following closely the procedure in (22), one reads a two-field variational

principle as

$$\delta\mathcal{L}_6\{u_i,\phi\} = \int_T dt\int_B \left[(\tau_{ij}+\tau^0_{ik}u_{j,k})_{,i}-\rho a_j\right]\delta u_j dV$$
$$+\int_T dt\int_B D_{i,i}\,\delta\phi dV$$
$$+\int_T dt\int_{\partial B}\left[t^{\star}_j-n_i(\tau_{ij}+\tau^0_{ik}u_{j,k})\right]$$
$$\cdot\ \delta u_j dS$$
$$\int_T dt\int_{\partial B}(\sigma^{\star}-n_iD_i)\,\delta\phi dS=0 \qquad (61)$$

which has the divergence equations and the associated natural boundary conditions as follows.

$$L^0_{1j}=(\tau_{ij}+\ \tau^0_{ik}u_{j,k})_{,i}-\ \rho a_j=0 \qquad \text{in}\quad \bar{B}XT \qquad (62)$$

$$L_2\ =\ D_{i,i}=0 \qquad \text{in}\quad \bar{B}XT \qquad (63)$$

$$L^0_{1j}=\ t^{\star}_j\ -\ n_i(\tau_{ij}+\tau^0_{ij}u_{j,k})=0 \qquad \text{on}\ \partial BXT \qquad (64)$$

$$L^{\star}_2\ =\ (\sigma^{\star}\ -n_iD_i)=0 \qquad \text{on}\ \partial BXT \qquad (65)$$

as its Euler-Lagrange equations (cf. [61]).

The fundamental equations of piezoelectric strained media in the spatial state consist of the stress equations of motion or Cauchy's first law of motion (62) and Cauchy's second law of motion of the form

$$e_{ijk}\tau_{jk}=\ 0 \quad \text{in}\ \bar{B}XT \qquad (66)$$

the charge equation of electrostatics (63), the linearized version of strain-mechanical displacement relations (55), namely,

$$S_{ij}=e_{ij}=1/2(u_{i,j}+u_{j,i}) \qquad (67)$$

the charge equation of electrostatics (63), the quasi-static electric field-electric potential relations (56), the constitutive equations (59) and (60), the boundary conditions of surface charge and surface tractions (64) and (65), those of mechanical displacements and electric potential of the form

$$u_i-u^{\star}_i\ =\ 0 \qquad \text{on}\ \partial B_uXT \qquad (68)$$

$$\phi - \phi^{*} = 0 \qquad \text{on } \partial B_{\phi} XT \tag{69}$$

and the initial conditions of the form

$$u_i(x_j, t_o) - v_i^{*}(x_j) = 0,$$
$$\dot{u}_i(x_j, t_o) - w_i^{*}(x_j) = 0 \qquad \text{in } B(t_o) \tag{70}$$

$$\phi(x_i, t_o) - \psi^{*}(x_i) = 0 \qquad \text{in } B(t_o) \tag{71}$$

Of the fundamental equations, only (62)-(65) are included in the two-field variational principle (61) and the remaining fundamental equations and the usual condition (26) remain as the admissibility conditions of the principle. By paralleling the unified variational principle (45) in non-linear piezoelectricity, the variational principle can be further augmented so as to adjoin the remaining equations into (61). In doing so, the dislocation potentials of the form

$$\Delta_{11}^{0} = \int_B \lambda_{ij}^{0} \left[S_{ij} - 1/2(u_{i,j} + u_{j,i})\right] dV \tag{72a}$$

$$\Delta_{22}^{0} = \int_B \mu_i^{0} (E_i + \phi_{,i}) dV \tag{72b}$$

$$\Delta_{33}^{0} = \int_{\partial B_u} \left[\lambda_i^{0} (u_i - u_i^{*}) + \mu^{0} (\phi - \phi^{*})\right] dS \tag{72c}$$

are added to (53), namely,

$$\delta \mathcal{L}_7 = \delta \mathcal{L}_6 + \delta \int_T \Delta_{ii}^{0} dt = 0 \tag{73}$$

and by using the same approach as described in (53), the Lagrange multipliers are identified with

$$\lambda_{ij}^{0} = \tau_{ij}, \mu_i^{0} = -D_i, \lambda_j^{0} = t_j = n_i t_{ij}, \mu^{0} = -\sigma = -n_i D_i \tag{74}$$

Hence, upon substituting (74) into (73)and on bearing in mind the usual admissibility conditions, an augmented variational principle is obtained in the form

$$\delta \mathcal{L}\{u_i, e_{ij}, \tau_{ij}, t_i; \phi, \sigma, E_i, D_i\}$$
$$= \int_T dt \int_B (L_{1j}^{0} \delta u_j + L_{1j}^{0} \delta e_{ij} + K_{ij}^{0} \delta \tau_{ij}) dV$$
$$+ \int_T dt \int_B (L_2 \delta \phi + L_i \delta E_i + K_i \delta D_i) dV$$

$$+\int_T dt\int_{\partial B_t} L_{1j}^{0*}\delta u_j dS$$

$$+\int_T dt\int_{\partial B_u} L_i^{0*}\delta t_i dS +\int_T dt\int_{\partial B_\sigma} L_2^{*}\delta\phi\ dS$$

$$+\int_T dt\int_{\partial B_\phi} K^{*}\delta\sigma\ dS =0 \tag{75}$$

with the definitions (62)-(65) and

$$L_{ij}^0=\tau_{ij}-(C_{ijkl}S_{kl}-C_{ijk}E_k)$$

$$K_{ij}^0= e_{ij}-1/2(u_{i,j}+u_{j,i})$$

$$L_i^{0*}= u_i-u_i^* \tag{76}$$

and

$$L_i=D_i-(C_{ijk}S_{jk}+C_{ij}E_j)$$

$$K_i= -(E_i+\phi_{,i})$$

$$K^*= (\phi-\phi^*) \tag{77}$$

Under the usual admissibility conditions mentioned for (45), the variational principle yields the fundamental equations of piezoelectric strained media as its Euler-Lagrange equations, and conversely the principle is satisfied if the fundamental equations are met. The variational principle that is not found in the open literature covers the variational principles of linear piezoelectricity whenever the terms involving initial stresses are dropped out.

7- CONCLUSION

This paper presents certain integral and differential types of variational principles so as to generate, as their Euler-Lagrange equations, the fundamental equations of an electro-elastic solid with small piezoelectric coupling. The variational principles are deduced from Hamilton's principle by modifying it through Friedrichs's transformation under the usual continuity and differentiability conditions of field variables. The first variational principle $\delta\mathcal{L}_1\{u_i,\phi\}=0$ of (25) is a two-field principle that generates the divergence equations and the associated natural boundary conditions of

electroelastic solid. This variational principle is extended by use of the dislocation potentials and the Lagrange undetermined multipliers, and hence the variational principle of $\delta\mathcal{L}_2\{u_i^{(\alpha)}, \tau_{ij}^{(\alpha)}, t_{ij}^{(\alpha)}; \phi^{(\alpha)}, D_i^{(\alpha)}\}=0$ of (37) is formulated for a finite and bounded region of the electroelastic solid, with an internal surface of discontinuity. It is shown that the divergence equations and the associated natural boundary conditions for each region and the jump conditions across the surface of discontinuity form a set of necessary and sufficient conditions for the variational principle (37). Another extension of (25) by Friedrichs's transformations is the integral type of variational principle $\delta\mathcal{L}_3\{\Lambda_3\}=0$ of (46),this variational principle has, of course, all the features of classical or true variational principles and it produces all the fundamental equations of nonlinear electroelastic solid but Cauchy's second law of motion (1c) and the initial conditions (14) and (15) for free and independent variations of the admissible state $\Lambda_3 = \{u_i, S_{ij}, \tau_{ij}, t_i, \phi, \sigma, E_i, D_i\}$. In (46), introducing the complementary electric enthalpy $\mathcal{H}(\tau_{ij}, D_i)$ of (47) through the Legendre transformation of the electric enthalpy $H(S_{ij}, E_i)$, the variational principle $\delta\mathcal{L}_4\{u_i, \tau_{ij}, t_i; \phi, \sigma, D_i\}$ of (48) is formulated. This integral type of variational principle leads, as its Euler-Lagrange equations, to the inverted constitutive equations (49) in addition to the divergence equations and the associated natural boundary conditions. The variational principle $\delta\mathcal{L}_5\{S_{ij}, \tau_{ij}, t_i; E_i, \sigma, D_i\}=0$ of (52) is the reciprocal of $\delta\mathcal{L}_1\{\Lambda_1\}=0$ of (25), and it is readily extracted from (46). Moreover, two variational principles are derived for the incremental motions of a piezoelectric solid subjected to initial stresses. The variational principle $\delta\mathcal{L}_6\{u_i, \phi\}=0$ of (61) is precisely the counterpart of $\delta\mathcal{L}_1\{\Lambda_1\}=0$ of (25), and the variational principle $\delta\mathcal{L}_7\{u_i, e_{ij}, \tau_{ij}, t_i; \phi, \sigma, E_i, D_i\}=0$ of (75) is that of $\delta\mathcal{L}_3\{\Lambda_3\}=0$ of (46) in the case of piezoelectric strained solid.

The variational principles $\delta\mathcal{L}_3\{\Lambda_3\}$ of (46) and $\delta\mathcal{L}_7\{\Lambda_7\}=0$ of (75) are quite general, and they are compatible with and contain, as particular cases, some of variational principles (e.g., [21]-[27],[30]-[32],[47]-[49] and references therein) in the absence of elastic nonlinearities and/or initial stresses. Besides, the variational principles recover, of course, their counterparts in elastodynamics, if the terms involving the quasi-static electric field are dropped out

(cf. [5], [33], [34]). On the other hand, the unified variational principle (46) can be specialized to obtain a number of variational principles operating on some of the field variables. Of these variational principles, it is worthy to mention a two-field variational principle operating on the stresses and the electric displacements in the form

$$\delta \mathcal{L}_8\{\tau_{ij}, D_i\} = \int_T dt \int_B (K_{ij}\delta\tau_{ij} + K_i \delta D_i) dV = 0 \tag{78}$$

which holds if and only if the strain-mechanical displacement relations (3), the electric field-electric potential relations (4) and the electric boundary conditions (13) are satisfied. A three-field variational principle is recorded in the form

$$\begin{aligned}\delta \mathcal{L}_9\{u_i, \tau_{ij}, \phi\} &= \int_T dt \int_B (L_{1j}\delta u_j + L_2 \delta\phi + K_{ij}\delta\tau_{ij}) dV \\ &\quad + \int_T dt \int_{\partial B} \overset{\star}{L}_{1j}\delta u_j dS + \int_T dt \int_{\partial B} \overset{\star}{L}_2 \delta\phi dS = 0\end{aligned} \tag{79}$$

which operates on the mechanical displacements, the electric potential and the stresses and holds only for the case when the divergence equations (1) and (2), the strain-mechanical displacement relations (3) and the surface charge and traction boundary conditions (10) and (12) are met. Along this line, a four-field variational principle is expressed by

$$\begin{aligned}\delta \mathcal{L}_{10}\{\Lambda_{10}\} &= \int_T dt \int_B (L_{1j}\delta u_j + L_2\delta\phi + L_{ij}\delta S_{ij} \\ &\quad + L_i \delta E_i) dV \\ &\quad + \int_T dt \int_{\partial B} \overset{\star}{L}_{1j}\delta u_j dS \\ &\quad + \int_T dt \int_{\partial B} \overset{\star}{L}_2 \delta\phi dS = 0\end{aligned} \tag{80}$$

for all the admissible states $\Lambda_{10} = \{u_i, S_{ij}; \phi, E_i\}$ in the notation of (25) and (43); this operates on the mechanical displacements, the Lagrangian strain, the electric potential and the electric field, and it is subjected to the constraint conditions (3), (4), (11), and (13)-(15). Moreover, another variational principle readily follows from the variational principle (75) as

$$
\begin{aligned}
\delta \mathcal{L}_{11}\{e_{ij}, \tau_{ij}, t_i, E_i, \sigma, D_i\} \\
&= \int_T dt \int_B (L^0_{ij}\,\delta e_{ij} + K^0_{ij}\,\delta \tau_{ij})\,dV \\
&+ \int_T dt \int_B (L_i\,\delta E_i + K_i\,\delta D_i)\,dV \\
&+ \int_T dt \int_{\partial B} L^{0*}_i\,\delta t_i\,dS \\
&+ \int_T dt \int_{\partial B} K^{*}\,\delta_\sigma\,dS = 0 \qquad (81)
\end{aligned}
$$

in the notation of (76) and (77); this is obviously the reciprocal of the variational principle of (61), namely,

$$\delta \mathcal{L}_6 \{u_i, \phi\} = 0 \qquad (82)$$

In summary, the foregoing variational principles are grouped in two families dealing with the nonlinear and incremental motions of piezoelectricity. They provide a standard basis for generating approximate direct solutions in terms of the trial functions which can be readily chosen, by means of the finite element and variational methods. The unified variational principles (46) and (75) allow one to make simultaneous approximation upon all the field variables since their constraint conditions are removed and the unknown Lagrange multipliers are expressed in terms of original field variables, providing their physical interpretation. The admissible states of unified variational principles are subjected only to the initial conditions together with the usual continuity and differentiability conditions of field variables and the symmetry of stress tensor. With the exception of the latter condition, the inclusion of which is not studied yet, the removal of constraint conditions is of certain value from the standpoint of computational economy, and hence it is most often desirable. However, if the constraint conditions are satisfied by the trial functions, the choice of which is often burdensome, the approximation of some field variables becomes more accurate. On the other hand, by further application of Friedrichs's transformation, there is no difficulty in extending the present principles to the variational principles in which the thermal, polar and nonlocal effects as well as the probabilistic and relativistic ones and alike are incorporated (e.g., [62]-[65]). In addition, noteworthy is the removal of the initial conditions for the two families of variational principles as was described by Tiersten [18]. The work devoted to some of these cases and, in particular, that to illustrate the

use of the variational principles presented are now in progress, and they will be reported in a forthcoming memoir.

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CHAPTER 3

NONLINEAR ELECTROELASTIC EQUATIONS OF WAVE PROPAGATION AND VIBRATIONS IN QUARTZ BARS

ABSTRACT

This paper presents the nonlinear electroelastic equations of wave propagation and vibrations in a quartz bar of uniform cross-section. To begin with, Hamilton's principle is stated for a nonlinear elastic continuum with small piezoelectric coupling, and then by carrying out the pertinent variations, a variational principle with certain constraints is formulated. The constraints are incorporated into this principle through the dislocation potentials and Lagrange undetermined multipliers, and hence a generalized variational principle is derived for the motions of nonlinear piezoelectric continuum. Next, the generalized principle together with a series expansions of its mechanical displacements and electric potential, a system of nonlinear equations of the quartz bar is consistently obtained. These one-dimensional equations of higher orders of approximation in which account is taken of only the elastic nonlinearities govern all the types of extensional, flexural and torsional motions of quartz bar. Also, special motions of quartz bar and those of quartz bar with initial stresses are pointed out. Lastly, the fully linearized governing equations of quartz bar are considered, the uniqueness of their solutions is examined and the sufficient conditions are enumerated for the uniqueness.

1- INTRODUCTION

Essentially, piezoelectricity is a reversible, inherently anisotropic and polarizable but not magnetizable field, and the field is quasi-static and linear. In piezoelectricity, the elastic field is considered to be dynamic, while the electric field is taken to be static; this quasi-static approximation is well justified in all cases of engineering interest. Besides linearity in piezoelectricity, there may exist either an intrinsic nonlinearity or an induced nonlinearity. The former is peculiar to a piezoelectric material, whereas the latter is due to its deformation. The application of intrinsic or induced nonlinearity and/or both of them can significantly affect the mechanical behavior of piezoelectric elements. This is desirable in some cases an it has been examined only for a few particular

motions [1]. In view of this review article [1], the present paper is concerned in deriving the one-dimensional nonlinear electroelastic equations describing all the types of motion of thin cylindrical quartz bars.

Recently, extensive studies have been made of one-dimensional piezoelectric problems at low frequencies [2-5, 10-21]. They have been directed toward either deriving differential governing equations of bars as in few cases [2-5, 7-9] or solutions of specific bar problems as in most cases [10-21] Among the former cases, Milsom and his colleagues [2,3] have presented a three-dimensional mode-matching theory of piezoelectric rectangular quartz bar. Tiersten and Ballato [4] have constructed the macroscopic equations accounting for the nonlinear extensional motion of thin piezoelectric rods, and they have applied these equations in the analyses of both intermodulation and nonlinear resonance of quartz rods. As a special case of their electromagnetic theory of rods, Green and Naghdi [5] have studied the isothermal vibrations of piezoelectric crystal rods. Moreover, following Mindlin [6], the author [7-9] has derived a one-dimensional theory of vibrations, which accommodates all the types of motions of piezoelectric crystal bars for both low as well as high frequencies. He has taken into account all the mechanical and electrical effects, and also he has described an application to biomechanics.

Efforts to solve certain problems of piezoelectric bars have been recently made by various authors [1,10-21]. Eer Nisse [13] has calculated approximately the electrode stress effects for length-extensional and flexural resonant vibrations of long, thin bars of quartz. An analysis of the flexural-mode equation has been presented for a rod with a vibration isolator [14]. The mechanical behavior of a piezoelectric bar has been studied with an electrical voltage as well as a time-dependent flux of heat at one end [15]. A simple one-dimensional model has been used to investigate the effect of the relaxation time on the behavior of a semi-infinite piezoelectric rod under a thermal shock at its end [16]. Moreover, the extensional vibration of a cylindrical rod with longitudinal piezoelectric coupling has been dealt with in an approximate procedure, and the depolarizing-field effect has been analyzed in a rod of finite and infinite

length [17]. A detailed numerical analysis of the dispersion relations has been reported for the axisymmetric normal waves of a piezoelectrically active bar wavequide [18]. Further the vibrational dissipation characteristics of a piezoceramic bar have been considered [19], as has the electrical excitation of an asymmetrically radiating bar [20]. Most recently, Solov'ev [21] has examined the influence of the electroted zone on the natural frequency of thickness resonance of a piezoceramic rod of rectangular cross-section under the conditions of plain strain.

Our aim in the present paper is (i) to obtain variational formulation for the nonlinear equations o_ an electroelastic solid with small piezoelectric coupling, with the help of this formulation, (ii) to derive a one-dimensional nonlinear electroelastic equations describing all the types of motions of thin quartz rods, and then (iii) to consider special motions of quartz bars and those of quartz bars with initial stresses, and also to examine the uniqueness of solutions in the linearized bar equations.

In the description of motions of the electroelastic solid, only the elastic nonlinearities are included, and hence the electrical behavior is taken to be linear. Accordingly, in the treatment of quartz rods which have small piezoelectric coupling, the nonlinear stress equations of motion, the associated nonlinear boundary conditions and the nonlinear strain-mechanical displacement relations are used, while the linear charge equations of electrostatics, the associated linear boundary conditions and alike are employed. Also, in the constitutive equations, the second-order, third-order and fourth-order elastic coefficients of quartz are retained for the stress tensor, and only the linear terms for the electric displacements.

Specifically, the content of this paper is as follows. First, the three-dimensional nonlinear equations of electroelastic solid are summarized in Section 2. This is followed in Section 3, by Hamilton's principle for the electroelastic solid and the associated quasi-variational principles. The geometry of a quartz bar is described, and also the series expansion for the mechanical displacements and the electric potential of quartz bar are recorded in Section 4. The nonlinear electroelastic equations of quartz bar are derived

by means of the quasi-variational principles together with the series expansions in Section 5. Special motions of quartz bar are considered, and especially the linearized equations and the uniqueness in their solutions are studied in Section 6. Finally, the concluding remarks and further needs of research are indicated in Section 7.

N o t a t i o n - In this paper, standard Cartesian tensor notation is used in a Euclidean 3-space E. The x_k-system of the space E is identified with a fixed, right-handed sytem of Cartesian convected (intrinsic) coordinates. Einstein's summation convention is implied for all repeated Latin indices (1,2,3) and Greek indices (1,2), unless indices are enclosed with parantheses. Further, commas and primes stand for partial differentiations with respect to the indicated space coordinates and the coordinate x_3, the bar axis, respectively, and superposed dots for time differentiations. Asterisks are used to designate prescribed quantities. The symbol B(t) refers to a region B with its boundary surface ∂B and closure $\bar{B}(=B\cup\partial B)$, at time t in the space E, and BXT refers to the Cartesian product of the region B and the time interval $T=[t_o,t_1)$. Also, boldface brackets are introduced so as to denote the jump of enclosed quantity across a surface of discontinuity S of the region B.

2 - NONLINEAR PIEZOELECTRIC EQUATIONS

In the three-dimensional space E, let $B+\partial B$ stand for an arbitrary, simply-connected, finite and bounded region of space occupied by an anisotropic elastic continuum with small piezoelectric coupling at time $t=t_o$. The regular boundary surface ∂B is consist of the complementary subsurfaces (S_u,S_t) and $(S_\sigma,S_{\dagger})$, that is, $S_u\cup S_t=S_\sigma\cup S_{\dagger}=\partial B$ and $S_u\cap S_t=S_\sigma\cap S_{\dagger}=\emptyset$. Also, let $\bar{B}XT$ represent the domain of definitions for the functions of (x_k,t).

Now, the three-dimensional differential equations are expressed for the electroelastic continuum with small piezoelectric coupling in the x_k-system of Cartesian coordinates as follows [22,23].

Divergence Equations

$$t_{kl,k}-\rho a_l = 0 \quad \text{in} \quad \bar{B}\times T \tag{1}$$

$$t_{kl} = \tau_{kl}+T_{kl}=\tau_{kr}(\delta_{lr}+u_{l,r}) \tag{2}$$

$$D_{k,k}=0 \quad \text{in} \quad \bar{B}\times T \tag{3}$$

with the definitions

t_{kl} = asymmetric Lagrangian stress tensor measured per unit area of the undeformed body

τ_{kl} = symmetric Kirchhoff stress tensor

$T_{kl}=\tau_{kr}u_{l,r}$ = Maxwell electrostatic stress tensor

ρ = density of the undeformed body

a_k = Lagrangian acceleration vector

u_k = displacement vector

δ_{kl} = Kronecker delta

D_k = electric displacement vector

Here, Eq.(1) stands for the nonlinear stress equations of motion and Eq.(3) for the linear charge equation of electrostatics. In Eqs (1) and (3), when the stress tensor t_k per unit area of the undeformed body, associated with a surface in the deformed body, is referred to the base vectors in the deformed body, τ_{kl} arises, while if t_k is referred to the base vectors in the undeformed body, t_{kl} ensues.

Gradient Equations

$$S_{kl}= \frac{1}{2}(u_{k,l}+u_{l,k}+u_{r,k}u_{r,l}) \quad \text{in} \quad \bar{B}XT \tag{4a}$$

$$S_{kl}=e_{kl}+ \frac{1}{2}(e_{rk}+w_{rk})(e_{rl}+w_{rl}) \tag{4b}$$

$$e_{kl}= \frac{1}{2}(u_{k,l}+u_{l,k}), \quad w_{kl}=\frac{1}{2}(u_{k,l}-u_{l,k}) \tag{5}$$

$$E_k = -\phi_k \quad \text{in} \quad \bar{B}\times T \tag{6}$$

with the definitions

S_{kl} = Lagrangian strain tensor

e_{kl} = linear strain tensor

w_{kl} = rotation tensor

ϕ_k = electric potential

E_k = quasi-static electric field vector

Equation (4) represents the nonlinear strain-mechanical displacement relations and Eq. (5) the electric field-electric potential relations.

C o n s t i t u t i v e   E q u a t i o n s

$$\tau_{kl} = \frac{1}{2}\left(\frac{\partial H}{\partial S_{kl}} + \frac{\partial H}{\partial S_{lk}}\right) \quad \text{in } \bar{B}\times T \tag{7}$$

$$D_k = -\frac{\partial H}{\partial E_k} \quad \text{in } \bar{B}\times T \tag{8}$$

with dhe definitions

$H = U - E_k D_k$ = electric enthalpy

U = potential energy density

A quartic form of the electric enthalpy is recorded in the form

$$H = \frac{1}{2} C_{klmn}S_{kl}S_{mn} - \frac{1}{2} C_{kl}E_kE_l - C_{klm}E_kS_{lm} + \frac{1}{6} C_{klmnrs}S_{kl}S_{mn}S_{rs} + \frac{1}{24} C_{klmnrstu}S_{kl}S_{mn}S_{rs}S_{tu} \tag{9}$$

In view of Eqs. (7) and (8), this equation yields the nonlinear constitutive equations for the components τ_{kl} of the symmetric stress tensor and the linear constitutive equations for the components D_k of the electric displacement vector as

$$\tau_{kl} = C_{klmn}S_{mn} - C_{mkl}E_m + \frac{1}{2} C_{klmnrt}S_{mn}S_{rt} + \frac{1}{6} C_{klmnrtpu}S_{mn}S_{rt}S_{pu} \tag{10}$$

$$D_k = C_{klm}S_{lm} + C_{kl}E_l \quad \text{in } \bar{B}\times T \tag{11}$$

Here, C_{klmn}, C_{klmnrt} and $C_{klmnrtpu}$ are the second-order,

third-order and fourth-order elastic constants, C_{klm} is the piezoelectric strain constants and C_{kl} the dielectric permittivity. Of these constants, the elastic constants refer to free constants since they describe the stress-strain relations when the electric field is absent, while the remaining constants refer to clamped constants [24]. Further, the usual symmetry relations hold for these material constants, namely,

$$C_{klmn} = C_{lkmn} = C_{mnkl},\ C_{klm} = C_{kml},\ C_{kl} = C_{lk}$$

$$C_{klmnrt} = C_{mnklrt} = C_{rtmnkl} = C_{lkmnrt} \tag{12}$$

$$C_{klmnrtpu} = C_{mnklrtpu} = C_{rtmnklpu} = C_{pumnrtkl} = C_{lkmnrtpu}$$

B o u n d a r y   C o n d i t i o n s

$$t_k^* - n_l t_{lk} = t_k^* - n_l \tau_{lr}(\delta_{kr} + u_{k,r}) = 0 \quad \text{on} \quad S_t xT \tag{13}$$

$$\sigma^* - n_k D_k = 0 \quad \text{on} \quad S_\sigma xT \tag{14}$$

$$u_k - u_k^* = 0 \quad \text{on} \quad S_u xT \tag{15}$$

$$\phi - \phi^* = 0 \quad \text{on} \quad S_\phi xT \tag{16}$$

with the definitions,

$t_k = n_l t_{lk}$ = stress vector

n_k = outward unit vector normal to ∂B

$\sigma = n_k D_k$ = surface charge

I n i t i a l   C o n d i t i o n s

$$u_k(x_l, t_o) - u_k^*(x_l) = 0$$

$$\dot{u}_k(x_l, t_o) - w_k^*(x_l) = 0 \quad \text{in} \quad B(t_o) \tag{17}$$

$$\phi(x_k, t_o) - \psi^*(x_k) = 0$$

J u m p   C o n d i t i o n s

$$n_k [\![t_k]\!] = 0 \tag{18a}$$

on SXT

$$n_k [\![\tau_{kr}(\delta_{lr} + u_{l,r})]\!] + t_l^a = 0 \tag{18b}$$

$$n_k [D_k] = Q \qquad \text{on} \quad S \times T \tag{19}$$

$$[u_k] = 0 \tag{20}$$

$$[\phi] = 0 \tag{21}$$

with the definitions

t_1^a = applied prescribed surface traction

Q = electric surface charge density

S = material surface of discontinuity

Equations (1) - (17) completely describe the nonlinear behavior of electroelastic continuum with small piezoelectric coupling, and the last four equations arise at a material surface of discontinuity.

3- VARIATIONAL FORMULATION

In piezoelectricity, the fundamental equations have been often expressed in variational forms as the appropriate Euler equations of variational principles [1,25-30]. These variational principles have been primarily derived with the aid of Hamilton's principle [1,6,25], and they allow the establishment of lower order theories and approximate direct solutions in piezoelectricity [26,31]. Now, Hamilton's principle is stated for the nonlinear elastic continuum with small piezoelectric coupling as

$$\delta \mathcal{H} = \delta \int_T \mathcal{L}\, dt + \int_T \delta W dt = 0 \tag{22}$$

with the definitions

$$\mathcal{L} = \int_{\partial B} [K - H(S_{kl}, E_k)]\, dV \tag{23}$$

$$K = \frac{1}{2} \rho \dot{u}_k \dot{u}_k \tag{24}$$

$$\delta W = \int_B (t_k^* \delta u_k - \sigma^* \delta \phi)\, dS \tag{25}$$

where $\mathcal{L}$ is the Lagrangian function, K the kinetic energy density and δW the virtual work per unit area done by the prescribed surface tractions t_k^* and surface charge σ^*.

By inserting Eqs. (23)-(25) into Eq. (22), one arrives at the variational equation of the form.

$$\delta\mathcal{H} = \delta \int_T dt \int_B \left[\frac{1}{2} \rho \dot{u}_k \dot{u}_k - H(S_{kl}, E_k) \right] dV + \int_{\partial B} (t_t^* \delta u_k - \sigma^* \delta \Phi) dS) = 0 \tag{26}$$

where all variations vanish at $t=t_o$ and $t=t_1$. Taking the indicated variations, utilizing the fact that the operation of variation commutes with that of differentiation, integrating by parts with respect to time and employing the constitutive relations (7) and (8) and the constraints on the variations, Eq. (26) takes the form

$$\delta\mathcal{H} = -\int_T dt \int_B (\rho a_k \delta u_k + \tau_{kl} \delta S_{kl} - D_k \delta E_k) dV + \int_T dt \int_{\partial} (t_k^* \delta u_k - \sigma^* \delta\Phi) \; dS = 0$$

By substituting the nonlinear strain-mechanical displacement relations (4) and the linear electric field-electric potential relations into this equation, employing the divergence theorem and rearranging terms, one finally obtains

$$\begin{aligned} \delta\mathcal{H} = & \int_T dt \int_B \{ [\tau_{kr}(\delta_{lr} + u_{l,r})]_{,k} - \rho a_l \} \; \delta u_l dV \\ & + \int_T dt \int_B D_{k,k} \delta\Phi dV \\ & + \int_T dt \int_{\partial B} [t_k^* - n_l \tau_{lr}(\delta_{kr} + u_{k,r})] \; \delta u_k dS \\ & + \int_T dt \int_{\partial B} (\sigma^* - n_k D_k) \delta\Phi dS = 0 \end{aligned} \tag{27}$$

In deriving this variational principle, the principle of conservation of mass is considered and the condition

$$\delta u_k = \delta\Phi = 0 \quad \text{in } B(t_o) \text{ and } B(t_1) \tag{28}$$

is imposed. Since the variations δu_k and $\delta\Phi$ of the admissible state $\Lambda_h = \{ u_k, \Phi \}$ in Eq. (27) are arbitrary and independent inside the volume B and on the boundary surface ∂B, one has the nonlinear stress equation of electrostatics (3) and the associated natural boundary conditions of tractions and surface charge (13) and (14) as the appropriate Euler equations of the variational principle (27). This is a two-field variational principle, and it contains some of earlier variational principles as special cases [1,25,32,33]. Further, it is of interest to note that this variational principle can be extracted from the principle of virtual work as well [30].

The differential variational principle (27) can be used, as usual, in solving approximately the boundary-value problems of nonlinear elastic continuum with small piezoelectric coupling, provided that the initial conditions (17) may be left out of account by a variety of numerical techniques [34,35]. Besides, any approximating solution must satisfy the rest of the fundamental equations (4) and (6)-(8) in accordance with Eq. (27) as well as the usual continuity and differentiability conditions of field quantities and the condition that the stress tensor be symmetric. This feature of Hamilton's principle has been discussed very thoroughly by Tiersten [36] and Gurtin [37]. However, the constraint conditions (4) and (6)-(8) can be relaxed through certain methods [37-39]. Of these methods, Friedrich's transformation [38] is used herein so as to remove the constraint conditions due to its versatility and wide use in the literature [39]. Accordingly, to adjoin the constraint conditions into the quasi-variational principle (27), the dislocation potentials $\Delta_{\alpha\beta}$ each constraints as a zero times a Lagrange multiplier, are introduced as

$$
\begin{aligned}
\Delta_{11} &= \lambda_{kl}\left[S_{kl} - \frac{1}{2}(u_{k,l} + u_{l,k} + u_{r,k}u_{r,l})\right] \\
\Delta_{12} &= \lambda_k(u_k - \overset{*}{u}_k) \\
\Delta_{21} &= \mu(\phi - \overset{*}{\phi}) \\
\Delta_{22} &= \mu_k(E_k + \phi_{,k})
\end{aligned}
\tag{29}
$$

and they are added to Eq. (22), namely,

$$
\delta J = \delta \mathcal{H} + \delta \int_T dt \left\{ \int_B (\Delta_{11} + \Delta_{22}) dV + \int_{S_u} \Delta_{12} dS + \int_{S_\phi} \Delta_{21} dS \right\} = 0 \tag{30}
$$

with the virtual work of the form

$$
\delta W = \int_{S_t} \overset{*}{t}_k \delta u_k dS - \int_{S_\sigma} \overset{*}{\sigma} \delta \phi \, dS \tag{31}
$$

Then, treating all the variations in Eq. (30) as free, one finds

$$\delta J = \int_T dt \int_B \left[-\rho a_k \delta u_k - \frac{1}{2}\left(\frac{\partial H}{\partial S_{kl}} + \frac{\partial H}{\partial S_{lk}}\right)\delta S_{kl} - \frac{\partial H}{\partial E_k} \delta E_k \right] dV$$
$$+ \int_T dt \int_{S_t} t_k^* \delta u_k - \int_T dt \int_{S_\sigma} \sigma^* \delta\Phi \, dS$$
$$+ \int_T dt \int_B \{ \delta\lambda_{kl} [S_{kl} - \frac{1}{2}(u_{k,l} + u_{l,k} + u_{r,k} u_{r,l})]$$
$$+ \lambda_{kl} [\delta S_{kl} - (\delta u_{k,l} + u_{r,k} \delta u_{r,l})]\} dV$$
$$+ \int_T dt \int_B [\delta\mu_k (E_k + \Phi_{,k}) + \mu_k (\delta E_k + \delta\Phi_{,k})] dV$$
$$+ \int_T dt \int_{S_u} [\delta\lambda_k (u_k - u_k^*) + \lambda_k \delta u_k] dS$$
$$+ \int_T dt \int_{S_\Phi} [\delta\mu (\Phi - \Phi^*) + \mu\delta\Phi] dS = 0 \qquad (32)$$

As before, by applying the divergence theorem and after some rearrangement, the stationary condition (32) readily gives the Lagrange multiplier in the form

$$\lambda_{kl} = \tau_{kl} \;, \; \mu_k = -D_k, \; \lambda_k = t_k = n_l t_{lk}$$
$$\mu = -\sigma = -n_k D_k \qquad (33)$$

since the volumetric variations δu_k, $\delta\Phi$, δS_{kl}, δE_k, $\delta\lambda_{kl}$ and $\delta\mu_k$ are arbitrary and independent in the region B and the surface variations δu_k, $\delta\Phi$, $\delta\lambda_k$ and $\delta\mu$ on the boundary surfaces S_u, S_Φ, S_t and S_σ.

Finally, from Eqs. (30), (31) and (33), one obtains the variational principle as follows.

$$\delta J\langle\Lambda\rangle = \int_T \delta J_{klkl} dt = 0 \qquad (34a)$$

where

$$\Lambda = \{u_k, t_k, \tau_{kl}, S_{kl}; \Phi, \sigma, D_k, E_k\} \qquad (34b)$$

and

$$\delta J_{1111} = \int_B \{ [\tau_{kr}(\delta_{lr} + u_{l,r})]_{,k} - \rho a_l \} \delta u_l dV \qquad (35)$$

$$\delta J_{1212} = \int_B D_{k,k} \delta\Phi dV \qquad (36)$$

$$\delta J_{1313} = \int_B \left[\tau_{kl} - \frac{1}{2}\left(\frac{\partial H}{\partial S_{kl}} + \frac{\partial H}{\partial S_{lk}}\right)\right]\delta S_{kl} dV \tag{37}$$

$$\delta J_{2121} = -\int_B \left(D_k + \frac{\delta H}{\delta E_k}\right)\delta E_k dV \tag{38}$$

$$\delta J_{2222} = \int_B \left[S_{kl} - \frac{1}{2}(u_{k,l} + u_{l,k} + u_{r,k}u_{r,l})\right]\delta\tau_{kl} dV \tag{39}$$

$$\delta J_{2323} = -\int_B (E_k + \Phi_{,k})\delta D_k dV \tag{40}$$

$$\delta J_{3131} = \int_{S_t} [t_k^* - n_l \tau_{lr}(\delta_{kr} + u_{k,r})]\delta u_k dS \tag{41}$$

$$\delta J_{3232} = \int_{S_\sigma} (\sigma^* - n_k D_k)\delta\Phi dS \tag{42}$$

$$\delta J_{3333} = -\int_{S_u} (u_k^* - u_k)\delta t_k dS - \int_{S_\Phi} (\Phi^* - \Phi)\,\delta\sigma dS \tag{43}$$

This variational principle may be written in a compact form by

$$\begin{aligned}\delta J\langle\Lambda\rangle = \delta\int_T dt\{\int_B\{\tau_{kl}&\left[S_{kl} - \frac{1}{2}(u_{k,l} + u_{l,k} + u_{r,k}u_{r,l})\right]\\ &- D_k(E_k + \Phi_{,k}) + K - H(S_{kl}, E_k)\}dV\\ &- \int_{S_u}(u_k^* - u_k)t_k dS + \int_{S_t} t_k^* u_k dS\\ &- \int_{S_\Phi}(\Phi^* - \Phi)\sigma dS + \int_{S_\sigma}\sigma^*\Phi dS\} = 0\end{aligned} \tag{44}$$

The variational equation (34) or (44) generates, as its Euler equations, the fundamental equations of nonlinear elastic continuum with small piezoelectric coupling, and hence we conclude a variational principle below.

V a r i a t i o n a l   P r i n c i p l e : Let $B+\partial B$ denote regular, finite and bounded region of the space E, with its piecewise smooth boundary surface $\partial B(=S_u \cup S_t = S_\Phi \cup S_\sigma$ and $S_u \cap S_t = S_\Phi \cap S_\sigma = \emptyset$ and its closure $\bar{B}(=B \cup \partial B)$. Then, of all the admissible states $\Lambda(=u_k, t_k, \tau_{kl}, S_{kl}; \Phi, \sigma, D_k, E_k)$ which satisfy the initial conditions (17) as well as the symmetry of stress tensor τ_{kl} and the usual continuity and differentiability

conditions of field variables; if and only if, that admissible state Λ which satisfies the nonlinear stress equations of motion (1), the linear charge equation of electrostatics (3), the nonlinear strain-mechanical displacement relations (4), the electric field-electric potential relations (6), the nonlinear constitutive equations (7) and (8), and the natural boundary conditions (13)-(16), is determined by the variational equation $\delta J<\Lambda> = 0$ of Eq. (44) as its appropriate Euler equations.

The variational principle (44) is believed to be first reported herein, and it does agree with and represents, as special cases, certain earlier variational principles operating only on some of the field variables [25,30,32,33]. By use of the fundamental lemma of the calculus of variations, the principle (34) or (44) leads readily to all the fundamental equations of piezoelectric continuum with small piezoelectric coupling, Eqs. (1), (3), (4) (6)-(8) and (13)-(16), but the initial conditions (17); conversely, if these equations are met, the principle is obviously satisfied. Further, the variational principle can be readily expressed, following Tiersten [36], in an augmented form which incorporates the initial conditions as well as the jump conditions (18)-(21); the result is a differential variational principle [29,30].

In closing, it is of interest to note that the expressions δJ_{1313} and δJ_{2121} in Eqs. (37) and (38) take the form

$$\begin{aligned}
\delta\bar{J}_{1313} &= \int_B \Big[\tau_{kl} - (C_{klmn}S_{mn} - C_{mkl}E_m \\
&\quad + \frac{1}{2} C_{klmnrt}S_{mn}S_{rt} \\
&\quad + \frac{1}{6} C_{klmnrtpu}S_{mn}S_{rt}S_{pu})\Big]\,\delta S_{kl}\,dV \\
\delta\bar{J}_{2121} &= -\int_B \Big[D_k - (C_{klm}S_{lm} + C_{kl}E_l)\Big]\,\delta E_k\,dV
\end{aligned} \tag{45}$$

in the case when the constitutive equations (10) and (11) are used in lieu of Eqs. (7) and (8).

4- EXPANSION IN SERIES

This section deals with the description of bar geometry, the method of reduction in deriving the electroelastic equations of quartz bar and the expansion in series for the field variables of quartz bar.

G e o m e t r y o f Q u a r t z B a r - Consider an initially selender quartz bar of uniform cross-section in the Euclidean 3-space E. The bar is referred to a system of right-handed Cartesian convected coordinates x_k. The axes x_α are selected as the principal axes of bar cross-section, and the axis x_3 is taken as the locus of centroids of bar cross-sections which is a straight line in the undeformed bar. The cross-section of bar A is bounded by a simply-connected Jordan curve C, that is, sufficiently smooth and non-intersecting. Moreover, by definition, one has the fundamental assumption of bars, $d/\ell \ll 1$, where d is the maximum diameter of cross-section and ℓ is the length of quartz bar. In addition to this, no singularities of any type is supposed to be present within the region of quartz bar. Thus, the bar is treated as a one-dimensional continuous model of a three-dimensional body.

M e t h o d o f R e d u c t i o n - The presence of electric field and material anisotropy makes it almost always compulsory the use of approximate lower order equations in investigating the dynamic characteristics of piezoelectric elements. Of the standard techniques [6,40-43], to reduce the three-dimensional equations of piezoelectricity into the lower order equations, Mindlin's method of reduction [6], is especially suitable and wide use in the literature [25,40-43], and it is used herein so as to construct the nonlinear electroelastic equations of quartz bar. This method of reduction rests entirely on the series expansions of field variables which are inserted in a pertinent variational principle which is then integrated with respect to one-or two-dimension.

E x p a n s i o n i n P o w e r S e r i e s - Under the usual existence, regularity and smoothness assumptions of bars and their fundamental assumptions, mentioned above, a set of shape functions $(\beta_{11}, \beta_{12}, \ldots \beta_{mn})$ is selected, and the shape functions are taken to be complete for a given field quantity in the bar region. Then the electric

potential and the displacement components are represented by

$$\{\phi, u_k\} = \sum_{m,n=0}^{N=\infty} \{\phi^{(m,n)}(x_3,t), u_k^{(m,n)}(x_3,t)\}\beta_{mn}(x_1,x_2) \quad (46)$$

Here, $\phi^{(m,n)}$ and $u_k^{(m,n)}$ are unknown <u>a priori</u> and independent functions of electric potential and mechanical displacements of order (m,n) to be determined, and the shape functions β_{mn} of order (m,n) can be selected to be any type of functions which is appropriate to the contour of cross-section and they are taken as a power series of the form

$$\beta_{mn}(x_\alpha) = x_1^m x_2^n \quad (47)$$

in the present analysis.

5- NONLINEAR BAR EQUATIONS

In this section, by means of the method of reduction described in Section 4, the system of one-dimensional, nonlinear electroelastic equations of quartz bar is consistently derived. To begin with, the series expansions (46) are inserted into the variational principle (34), the volume integrals are split into an area integral over a cross-section of, and a line integral along, the quartz bar, and then the integrations are performed. The resulting equations are recorded below in terms of various field quantities of higher orders which are now defined.

F i e l d Q u a n t i t i e s o f O r d e r (m,n) - The stress resultants of order (m,n):

$$T_{kl}^{(m,n)} = \int_A x_1^m x_2^n \tau_{kl} dA \quad (48)$$

$$N_k^{(m,n)} = \sum_{p+q+0}^{N} \{[mpT_{11}^{(m+p-2,\ n+q)}$$

$$+ (np+mq)T_{12}^{(m+p-1\ ,\ n+q-1)} + qnT_{22}^{(m+p,\ n+q-1)}$$

$$+ p\acute{T}_{31}^{(m+p-1,\ n+q)} + q\acute{T}_{32}^{(m+p,\ n+q-1)}]u_k^{(p,q)}$$

$$+ T_{33}^{(m+p,\ n+q)} u_k'' (p,q) + \Big[(p+m) T_{23}^{(m+p-1\ ,\ n+q)}$$

$$+ (q+n) T_{23}^{(m+p,\ n+q-1)} + T_{33}'^{(m+p,\ n+q-1)}\Big] u_k^{(p,q)}\}$$

the surface loads of order (m,n):

$$P_k^{(m,n)} = \oint_C x_1^m x_2^n \gamma_\alpha \tau_{\alpha k} dA \tag{49}$$

$$Q_k^{(m,n)} = P_k^{(m,n)} + R_k^{(m,n)} \tag{50}$$

$$R_k^{(m,n)} = \sum_{p+q=0}^{N} \Big[(pP_1^{(m+p-1,\ n+q)}$$

$$+ qP_2^{(m+p,\ n+q-1)}) u_k^{(p,q)} + P_3^{(m+p,\ n+q)} u_k^{(p,q)}\Big] \tag{51}$$

$$N_{3k}^{(m,n)} = \sum_{p+q=0}^{N} \Big[(pT_{31}(m+p-1,\ n+q)$$

$$+ qT_{32}^{(m+p,\ n+q-1)}) u_k^{(p,q)}$$

$$+ T_{33}^{(m+p,\ n+q)} u_k'^{(p,q)}\Big] \tag{52}$$

the acceleration of order (m,n) :

$$U_k^{(m,n)} = \sum_{p+q=0}^{N} I^{(m+p,\ n+q)} u_k^{(p,q)} \tag{53a}$$

$$A_k^{(m,n)} = \ddot{U}_k^{(m,n)} \tag{53b}$$

the prescribed stress resultants of order (m,n):

$$T_k^{*(m,n)} = \int_A x_1^m x_2^n t_k^* dA,\ P_k^{*(m,n)} = \oint_C x_1^m x_2^n t_k^* ds \tag{54}$$

the aerial moment of inertia of order (m,n):

$$I^{(m,n)} = \int_A x_1^m x_2^n dA \tag{55a}$$

with the usual quantities of bars as

$$I^{(0,0)} = A,\quad I^{(1,0)} = I^{(0,1)} = I^{(1,1)} = 0 \tag{55b}$$

the electric displacements of order (m,n):

$$D_k^{(m,n)} = \int_A x_1^m x_2^n D_k dA \tag{56}$$

the surface charge of order (m,n):

$$D^{(m,n)} = \oint_C x_1^m x_2^n \nu_\alpha D_\alpha ds \tag{57}$$

and the prescribed surface charge of order (m,n):

$$\Theta^{*(m,n)} = \int_A x_1^m x_2^n \sigma^* dA, \quad D^{*(m,n)} = \oint_C x_1^m x_2^n \sigma^* ds \tag{58}$$

are defined. In the above equations, ν_α denotes the unit outward vectors normal to the contour C of cross-section. Also, the electric enthalpy function G measured per unit length of the undeformed bar, namely,

$$G = \int_A H dA \tag{59}$$

is introduced for later convenience.

Equations of Motion - Consider the volume integral (3.14) of the form, namely,

$$\delta J_{1111} = \int_0^1 dx_3 \int_A \{ [\tau_{kr}(\delta_{1r} + u_{1,r})]_{,k} - \rho a_1 \} \delta u_1 dV \tag{60}$$

Substituting the series expansions of mechanical displacements (46) into this integral, carrying out the integrations over A, using the divergence theorem and replacing the stress and load resultants of order (m,n), one obtains

$$\delta J_{1111} = \int_L dx_3 \sum_{m+n=0}^{N} (T_{3k}^{(m,n)} - mT_{1k}^{(m-1,\ n)}$$

$$-nT_{2k}^{(m,\ n-1)} + N_k^{(m,n)} + Q_k^{(m,n)}$$

$$- \rho A_k^{(m,n)}) \delta u_k^{(m,n)} \tag{61}$$

where L stands for the interval $[0,1]$. When setting the variational equation (34) equal to zero for the arbitrary and independent variations of field quantities such as $\delta u_k^{(m,n)}$ in this case, one readily obtains the macroscopic equations of motion of order (m,n) in the form

$$T_{3k}^{\prime(m,n)} - mT_{1k}^{(m-1,\ n)} - mT_{2k}^{(m,\ n-1)} + N_k^{(m,n)}$$

$$+ Q_k^{(m,n)} - \rho A_k^{(m,n)} = 0 \quad \text{on} \quad L\text{x}T \tag{62}$$

for the quartz bar.

Charge Equation of Electrostatics As before, evaluating the volume integral J_{1212} of Eq. (36), one arrives at the macroscopic charge equation of electrostatics of order (m,n) in the form

$$D_3^{\prime(m,n)} - mD_1^{(m-1\ ,\ n)} - nD_2^{(m,\ n-1)} + D^{(m,n)} = 0 \tag{63}$$

in terms of the quantities defined by Eqs. (56) and (57)

Electric Field and Strain Distributions - Likewise, considering Eqs. (39) and (40), integrating over A and using the stress and electric displacements of order (m,n), the distribution of strain of order (m,n):

$$S_{kl}(x_m,t) = \sum_{m+n=0}^{N} x_1^{\,m} x_2^{\,n} S_{kl}^{(m,n)}(x_3,t) \tag{64}$$

where

$$S_{kl}^{(m,n)} = e_{kl}^{(m,n)} + \frac{1}{2}\sum_{p+q=0}^{m+n} (e_{rk}^{(m-p,\ n-q)}$$

$$+ w_{rk}^{(m-p,\ n-q)})(e_{rl}^{(p,q)} + w_{rl}^{(p,q)}) \tag{65}$$

with

$$e_{\alpha\beta}^{(m,n)} = \frac{1}{2}\Big[(m+1)(\delta_{1\alpha}u_\beta^{(m+1),\ n)} + \delta_{1\beta}u_\alpha^{(m+1,\ n)})$$

$$+ (n+1)\ (\delta_{2\alpha}u_\beta^{(m,\ n+1)} + \delta_{2\beta}u_\alpha^{(m,\ n+1)})\Big]$$

$$e_{\alpha 3}^{(m,n)} = \frac{1}{2}\left[u_{\alpha}^{\prime (m,n)} + (m+1)\delta_{1\alpha}u_3^{(m+1,\ n)} + (n+1)\delta_{2\alpha}u_3^{(m,\ n+1)}\right]$$

$$w_{\alpha\beta}^{(m,n)} = \frac{1}{2}\left[(m+1)(\delta_{1\beta}u_{\beta}^{(m+1,\ n)} - \delta_{1\alpha}u_{\beta}^{(m+1,\ n)}) + (n+1)(\delta_{2\beta}u_{\alpha}^{(m,\ n+1)} - \delta_{2\alpha}u_{\beta}^{(m,\ n+1)})\right]$$

$$w_{\alpha 3}^{(m,n)} = \frac{1}{2}\left[u_{\alpha}^{\prime (m,n)} - (m+1)\delta_{1\alpha}u_3^{(m+1,\ n)} - (n+1)\delta_{2\alpha}u_3^{(m,\ n+1)}\right]$$

$$e_{33}^{(m,n)} = u_3^{\prime (m,n)}, \quad w_{33}^{(m,n)} = 0 \tag{66}$$

and that of electric field of order (m,n):

$$E_k(x_1,t) = \sum_{m+n=0}^{N} x_1^{m} x_2^{n} E_k^{(m,n)}(x_3,t) \tag{67}$$

where

$$E_{\alpha}^{(m,n)} = -\left[(m+1)\delta_{1\alpha}\phi^{(m+1,\ n)} + (n+1)\delta_{2\alpha}\phi^{(m,\ n+1)}\right]$$

$$E_3^{(m+n)} = -\phi^{\prime}(m,n) \tag{68}$$

are found for the vanishing of the coefficients of free variations of the stress resultants and electric displacements of order (m,n) of quartz bar in the variational equation (34).

C o n s t i t u t i v e E q u a t i o n s - Paralleling to the derivation of electric field and strain distributions above, the volume integrals (37) and (38) are evaluated by use of Eqs. (48), (56), (64) and (67), and then the constitutive relations are obtained for the stress resultants of order (m,n) and the electric displacements of order (m,n) in the form.

$$T_{kl}^{(m,n)} = \frac{1}{2}\left(\frac{\partial G}{\partial S_{kl}^{(m,n)}} + \frac{\partial G}{\partial S_{lk}^{(m,n)}}\right) \text{ on } L\times T \tag{69}$$

$$D_k^{(m,n)} = - \frac{\partial G}{\partial E_k^{(m,n)}} \qquad \text{on} \quad L \times T \tag{70}$$

in terms of the electric enthalpy function G of Eq. (59)

In the case of the linear constitutive equations (10) and (11), the volume integrals (45) are evaluated in lieu of Eqs. (37) and (38) with the result,

$$\begin{aligned} T_{kl}^{(m,n)} &= C_{klrt} \sum_{a+b=0}^{N} I_{(m+a\ ,\ n+b)} S_{rt}^{(a,b)} \\ &- C_{rkl} \sum_{a+b=0}^{N} I_{(m+a,\ n+b)} E_r^{(a,b)} \\ &+ \frac{1}{2} C_{klrtpq} \sum_{a+b=0}^{N} \sum_{c+d=0}^{N} \chi S_{rt}^{(a,b)} S_{pq}^{(c,d)} \\ &+ \frac{1}{6} C_{klrtpquv} \sum_{a+b=0}^{N} \sum_{c+d=0}^{N} \sum_{e+f=0}^{N} \zeta S_{rt}^{(a,b)} S_{pq}^{(c,d)} S_{uv}^{(e,f)} \end{aligned} \tag{71}$$

with

$$\chi = I_{(m+a+c,\ n+b+d)},\ \zeta = I_{(m+a+c+e,\ n+b+d+f)}$$

and

$$\begin{aligned} D_k^{(m,n)} &= C_{klr} \sum_{a+b=0}^{N} I_{(m+a,\ n+b)} S_{lr}^{(a,b)} \\ &+ C_{kl} \sum_{a+b=0}^{N} I_{(m+a,\ n+b)} E_l^{(a,b)} \end{aligned} \tag{72}$$

in terms of $I_{(m,n)}$ of Eq. (5.8).

B o u n d a r y C o n d i t i o n s - The mechanical displacements and the surface charge are prescibed on the surface portion S_d of the lateral surface of bar S_1 and on the left face boundary surface A_1, while the traction and the electric potential are prescibed on the remaining portion S_τ of S_1 and on the right face boundary surface A_r, where $S_u = S_\sigma = S_d \cup A_1$, $S_t = S_\phi = S_\tau \cup A_r$, $S_d \cup S_\tau = S_1$, $A_1 \cup S_d \cup A_r = \partial B$. As in the derivation of the stress equations of motion and the charge equation of electrostatics, by evaluating the surface integrals (41)-(43) of the variational principle (34), the natural boundary

conditions are expressed for the tractions of order (m,n) by

$$P_k^{*(m,n)} - (P_k^{(m,n)} + R_k^{(m,n)}) = 0 \text{ on } S_d \times T \tag{73}$$

$$T_k^{*(m,n)} - (T_{3k}^{(m,n)} + N_{3k}^{(m,n)}) = 0 \text{ on } A_1 \times T \tag{74}$$

for the surface charge of order (m,n) by

$$D^{*(m,n)} - D^{(m,n)} = 0 \quad \text{on} \quad S_d \times T \tag{75}$$

$$\sigma^{*(m,n)} + D_3^{(m,n)} = 0 \quad \text{on } A_1 \times T \tag{76}$$

for the mechanical displacements of order (m,n) by

$$u_k^{*(m,n)} - u_k^{(m,n)} = 0 \quad \text{on } S_u \times T \tag{77}$$

and for the electric potential of order (m,n) by

$$\Phi^{*(m,n)} - \Phi^{(m,n)} = 0 \quad \text{on } S_\Phi \times T \tag{78}$$

Here, (t_k and Φ) and (u_k and σ) are prescribed, since they are the most commonly encountered in practice [7].

I n i t i a l   C o n d i t i o n s - By making use of Eq. (17) and Eq. (46), one reads the initial conditions of order (m,n) as

$$u_k^{(m,n)}(x_3,t) - \upsilon_k^{*(m,n)}(x_3) = 0 \text{ on } L(t_o) \tag{79}$$

$$\dot{u}_k^{(m,n)}(x_3,t) - w_k^{*(m,n)}(x_3) = 0 \text{ on } L(t_o) \tag{80}$$

and

$$\Phi^{(m,n)}(x_3,t) - \psi^{*(m,n)}(x_3) = 0 \quad \text{on } L(t_o) \tag{81}$$

where υ_k^*, w_k^* and ψ^* are given functions of x_3.

Thus far, the one-dimensional, nonlinear equations of successively higher orders of approximation are consistently developed for quartz bars on the basis of three-dimensional theory of piezoelectricity. These governing equations of order (m,n) consist of the electric potential and mechanical displacement fields (46), the stress equations of motion (62), the charge equation of electrostatics (63), the electric field and strain distributions (65) and (67), the constitutive equations (69) and (70) or (71) and (72), the natural

boundary conditions (73) - (78) and the initial conditions (79) - (81). The number of the governing equations is infinite, that is, $m+n=0,1,2,\ldots$, $N=\infty$, and hence the equations are not formally determinate yet; they will be made deterministic in the next section.

6- SPECIAL MOTIONS

To obtain a deterministic system of nonlinear electroelastic equations of quartz bar derived in the previous section, these infinite number of equations with their infinite number of unknowns must be consistently reduced to a finite number of equations with their finite number of unknowns by a process of series truncation. The process of truncation, special motions of quartz bar, and especially the linearized governing equations and the uniqueness in their sulotions are taken up in this section. Further, the motions of quartz bar with initial stresses are pointed out.

D e t e r m i n i s t i c B a r E q u a t i o n s - The foregoing derivation of the governing equations of quartz bar rests entirely on the fields of mechanical displacements and electric potential, chosen a priori as a starting point and representing them by the power series expansions (46) of which the terms $u_k^{(m,n)}$ and $\phi^{(m,n)}$ are already taken to be exist. Thus, the governing equations of order (M,N) is defined by either

$$\{\phi, u_k\} = \sum_{m=0}^{M} \sum_{n=0}^{N} x_1^m x_2^n \{\phi^{(m,n)}, u_k^{(m,n)}\} \tag{82a}$$

or the series expansions (46) together with the condition

$$\phi_k^{(m,n)} = u_k^{(m,n)} = 0 \text{ for all } m \geq M+1, \quad n \geq N+1 \tag{82b}$$

and only the quantities involved in (82) are kept in the equations. In view of Eqs. (82), there exists the $4(M+1).(N+1)$ unknowns $u_k^{(m,n)}$ and $\phi^{(m,n)}$ and equations to solve them. In addition to Eqs. (82), another type of deterministic governing equations is simply defined by

$$\phi^{(m,n)} = u_k^{(m,n)} = 0 \text{ for all } (m+n) \geq N+1 \tag{83}$$

where N is a positive integer. This obviously considers the same weight for both of the lateral coordinates x_1 and x_2.

Further, in both the definitions (82) and (83), by selecting the positive integers M and N or only N for particular applications, the governing equations incorporate as many higher order effects as deemed necessary. Hence the customary correction factors of bars are naturally abrogated [6].

S p e c i a l   M o t i o n s - Of the special motions of quartz bar, the extensional motions [44], can be examined by representing the electric potential and the mechanical displacements as in Eq. (46) with the condition $u_\alpha = u_\alpha(x_1, x_2, t) \cong 0$. Also, in the case of low-frequency extensional motions, it is appropriate to take the vanishing boundary stresses on the lateral boundary surface S_1, and hence all the vanishing stresses but T_{33}. The electrical boundary conditions depend on the surface S_ϕ, and if the edge boundary surfaces $S_e(=A_1 U A_r)$ are fully electroded, the boundary conditions become $D_\alpha = 0$ in Eqs. (75) and (57); this will be reported later [45]. Moreover, the governing equations of quartz bar can be specialized to study its nonlinear torsional motions in the sense of Saint-Venant by the use of the displacement field (46) together with the condition [46]

$$u_1 = x_2 u_1^{(0,1)}, \quad u_2 = x_1 u_2^{(1,0)}, \quad u_3^{(m,n)} = \omega C_{mn} \tag{83a}$$

and

$$u_1^{(0,1)} = - u_2^{(1,0)} = -\omega x_3 \tag{83b}$$

Here, $\omega = w_{12}$ denotes the uniform rate of twist and C_{mn} is a constant.

L i n e a r   B a r   E q u a t i o n s - Dropping out all the terms involving nonlinearity, namely,

$$W_{kl}^{(m,n)} = 0, \quad N_k^{(m,n)} = R_k^{(m,n)} = N_{3k}^{(m,n)} = 0$$

$$I_{(m+a+c,\ n+b+d)} = I_{(m+a+C+e,\ n+b+d+f)} = 0 \tag{84}$$

in the macroscopic electroelastic equations of Section 5, the fully linear governing equations of quartz bar are obtained. They are the macroscopic equations of motion as

$$T'^{(m,n)}_{3k} - mT_{1k}^{(m-1,\ n)} - nT_{2k}^{(m,\ n-1)} + P_k^{(m,n)}$$

$$- \rho A_k^{(m,n)} = 0 \quad \text{on} \quad L \times T \tag{85}$$

the associated boundary conditions of tractions as

$$p_k^{*(m,n)} - p_k^{(m,n)} = 0 \quad \text{on} \quad S_d \times T \tag{86}$$

$$T_k^{*(m,n)} - T_{3k}^{(m,n)} = 0 \quad \text{on} \quad A_1 \times T \tag{87}$$

the macroscopic charge equation of electrostatics (63), the distribution of electric field (67) and that of strain by

$$S_{kl}^{(m,n)} = e_{kl}^{(m,n)} \tag{88}$$

the constitutive equations for the gross electric displacements (72) and those for the stress resultants in the form

$$T_{kl}^{(m,n)} = \sum_{p+q=0}^{N} I_{(m+p,\ n+q)} (C_{klrt} S_{rt}^{(p,q)} - C_{rkl} E_r^{(p,q)}) \tag{89}$$

the boundary conditions of surface charge (75) and (76), those of mechanical displacements (77) and those of electric potential (78), and the initial conditions (79)-(81). The linear governing equations of quartz bar recover those by the author [7], who has employed a semi-variational approach in his derivation.

Uniqueness of Solutions - The solutions of an initial mixed-boundary value problem defined by the one-dimensional linear governing equations of quartz bar are shown to be unique by means of the logarithmic convexity arguments [47]. To establish this, as usual, the existence of two solutions arising from the same date $d_k^{(1)}$ and $d_k^{(2)}$ is supposed and the difference solution $d_k (= d_k^{(1)} - d_k^{(2)})$ is considered. The difference solution, that is, as before, $u_k (= u_k^{(1)} - u_k^{(2)})$ and $\phi (= \phi^{(1)} - \phi^{(2)})$ evidently satisfies the homogeneous parts of the governing equations by virtue of the linearity of these equations. Accordingly, it suffices to show that the difference solution is trivial for the homogeneous governing equations in proving the uniqueness of solutions. The treatment of uniqueness begins by defining the function F(t) by

$$\mathfrak{F}(t) = \log F(t), \quad t \in T \tag{90a}$$

$$F(t) = \frac{1}{2} \int_L dx_3 \int_A \rho u_k u_k dA \quad , \quad \tau_1 < t < \tau_2 \tag{90b}$$

$$F(t) = 0 \ , \quad t \varepsilon [t_o, \tau_1] \text{ and } t \varepsilon [\tau_2, t_1] \tag{90c}$$

Here, Eq. (90c) clearly implies the uniqueness for all $t \varepsilon T$ but $t \varepsilon |\tau_1, \tau_2|$; F(t) can be chosen, without loss of generality, as in this equation. Thus, only the interval $\tau = (\tau_1, \tau_2)$ is considered on which F(t) is positive by definition, and this function should satisfy the condition of the form

$$F^2 \ddot{\mathcal{F}} = F\ddot{F} - \dot{F}^2 \geq 0 \ , \quad \tau_1 < t < \tau_2 \tag{91}$$

for the convexity of $\mathcal{F}(t)$.

Now, the kinetic energy K, the internal energy W and the total energy Ω per unit length of the quartz bar are calculated in the form

$$K = \frac{1}{2} \int_A \rho \dot{u}_k \dot{u}_k = \frac{1}{2} \sum_{m+n=0}^{N} \rho \dot{U}_k^{(m,n)} \dot{u}_k^{(m,n)} \tag{92}$$

$$W = \frac{1}{2} \int_A (\tau_{kl} e_{kl} + E_k D_k) dA = \frac{1}{2} [T_{3k}^{((m,n)} u_k'^{(m,n)} + (mT_{1k}^{(m-1,\ n)} + nT_{2k}^{(m,\ n-1)}) u_k^{(m,n)} + D_3^{(m,n)} \phi'^{(m,n)} + (mD_1^{(m-1,n)} + nD_2^{(m,\ n-1)}) \phi] \tag{93}$$

$$\Omega = K + W \tag{94}$$

where the series expansions (46), the definitions (48), (53), (55) and (56) and the distributions (67) and (88) are used. Likewise, Eq. (90b) is evaluated with the result

$$F(t) = \frac{1}{2} \int_L \rho \sum_{m+n=0}^{N} U_k^{(m,n)} u_k^{(m,n)} dx_3 \tag{95}$$

in the interval τ. Then, time differentiations of this equation, by assuming the usual smoothness of functions, lead to

$$\dot{F}(t) = \int_L \rho \sum_{m+n=0}^{N} \dot{U}_k^{(m,n)} u_k^{(m,n)} dx_3 \tag{96}$$

and

$$\ddot{F}(t) = \int_L (2K + \rho \sum_{m+n=0}^{N} A_k^{(m,n)} u_k^{(m,n)}) dx_3 \tag{97}$$

in which Eqs. (53) and (92) are used. With the help of the homogeneous part of Eq. (85), Eq. (97) is expressed in the form

$$\ddot{F}(t) = \int_L \Big[2K + \sum_{m+n=0}^{N} \big(T_{3k}'^{(m,n)} - mT_{1k}^{(m-1,\,n)} - nT_{2k}^{(m,\,n-1)} + P_k^{(m,n)}\big) u_k^{(m,n)} \Big] dx_3 \tag{98}$$

Then, on combining Eqs. (93) and (94) and integrating by parts, Eq. (98) takes the form

$$\ddot{F}(t) = -2W + \int_L 4K dx_3 + \Gamma + X \tag{99}$$

with

$$\Gamma = \sum_{m+n=0}^{N} \big(T_{3k}^{(m,n)} u_k^{(m,n)} + D_3^{(m,n)} \phi^{(m,n)}\big)\Big|_{x_3=0}^{1} \tag{100}$$

$$X = \int_L \sum_{m+n=0}^{N} \big(P_k^{(m,n)} u_k^{(m,n)} + D^{(m,n)} \phi^{(m,n)}\big) dx_3 \tag{101}$$

where Eq. (63) is taken into account. By the conservation of energy and the initial conditions (79)-(81), the total energy Ω is equal to zero. Besides, the boundary conditions (75)-(78), (86) and (87) render Γ and X to zero, and then Eq. (99) becomes

$$\ddot{F}(t) = \int_L 4K dx_3 \tag{102}$$

In view of Eqs. (95), (96) and (102), one writes the right of Eq. (91) as

$$F^2 \ddot{\mathcal{F}} = \int_L \rho^2 \sum_{m+n=0}^{N} \big[U_k^{(m,n)} u_k^{(m,n)}\big] \cdot \big[\dot{U}_k^{(m,n)} \dot{u}_k^{(m,n)}\big] dx_3 - \Big[\int_L \rho \sum_{m+n=0}^{N} U_k^{(m,n)} u_k^{(m,n)} dx_3\Big]^2 \tag{103}$$

By virtue of Schwartz's inequality, one finds

$$F^2 \ddot{\mathcal{F}} \geq 0 \tag{104}$$

on the interval τ, and after integration, this implies

$$F(t) \leq [F(\tau_1)]^{\tau_2 - \tau / \tau_2 - \tau_1} [F(\tau_2)]^{t - \tau_1 / \tau_2 - \tau_1} \quad \text{on } \tau \tag{105}$$

Due to the continuity of $F(t)$, $F(\tau_1)=0$, Eq. (105) shows that $F(t)=0$ for the interval τ as well, contrary to the initial hypothesis $F(t)>0$. Hence $F(t)=0$ for all $t \in T$, and

the difference solution is trivial, that is, the uniqueness is established as in the case of polar rods, [48]. A theorem of uniqueness is concluded as follows.

T h e o r e m - Given a regular region of finite bar space $B+\partial B$ with its boundary surface $\partial B(=S_t \cup S_u = S_\sigma \cup S_\Phi$, $S_t \cap S_u = S_\sigma \cap S_\Phi = \emptyset)$ in the Euclidean 3-Space E, then there exists at most one set of twice continuously differentiable functions $u_k^{(m,n)}$ and $\Phi^{(m,n)}$ in $B+\partial B$ at the time interval T, obeying Eqs. (63), (67), (89), (72), (88) and (89), and satisfying the boundary conditions (75)-(78), (86) and (87) and the initial conditions (79)-(81).

Q u a r t z B a r W i t h I n i t i a l S t r e s s e s - In the x_k- fixed system of Cartesian convected coordinates, consider the piezoelectric medium $B+\partial B$ with its boundary surface ∂B and closure $\bar{B}$. The medium is under initial stresses in its reference (initial) state which is considered to be self-equilibrating following loading in the natural state of medium. Then a small motion is superimposed upon the reference state. For this motion, the set of fundamental equations is consist of the stress equations of motion (1) and the boundary conditions of tractions (13) with the condition [49,50]:

$$\tau_{kl} = \overset{o}{\tau}_{kr} u_{l,r} \qquad \text{in } \bar{B}\text{xT} \qquad (106)$$

or

$$(\tau_{kl} + \overset{o}{\tau}_{kr} u_{l,r})_{,k} - \rho_o a_l = 0 \quad \text{in } \bar{B}\text{xT} \qquad (107)$$

$$t_k^* - n_l(\tau_{lk} + \overset{o}{\tau}_{lr} u_{k,r}) = 0 \qquad \text{on } S_t\text{xT} \qquad (108)$$

the charge equation of electrostatics (3), the strain-mechanical displacement relations:

$$S_{kl} = e_{kl} = \frac{1}{2}(u_{l,k} + u_{l,k}) \quad \text{in } \bar{B}\text{xT} \qquad (109)$$

the electric field-electric potential relations (6), the constitutive relations

$$\tau_{kl} = C_{klmn} S_{mn} - C_{mkl} E_m \qquad (110)$$

and Eq. (11), the boundary conditions of dislacements, surface charge and electric potential (14)-(16) and the initial conditions (17) in the spatial (final) state.

In the above equations; τ_{kl}, u_k, a_k and so on indicate small incremental quantities superimposed upon those of the reference state denoted (o) such as ($\tau^o_{kl}, u^o_k, t^o_k$). The incremental components of displacements u_k and the electric potential ϕ are represented by the series expansions (46). By paralleling to the derivation in Section 5, the macroscopic equations of thin quartz bar with initial stresses may be established by means of a variational principle [28] and the series expansions (46) as

$$T_{3k}^{\prime(m,n)} - mT_{1k}^{(m-1,\ n)} - nT_{2k}^{(m,\ n-1)} + P_k^{(m,n)}$$

$$+ N_k^{o(m,n)} + R_k^{o(m,n)} - \rho A_k^{(m,n)} = 0 \quad \text{on} \quad L\times T$$

$$T_k^{*(m,n)} - (T_{3k}^{(m,n)} + N_{3k}^{o\ (m,n)}) = 0 \quad \text{on} \ A_1 \times T \qquad (111)$$

with the definitions (48) and (50)-(52) in terms of the incremental quantities, and alike [51].

7- CONCLUSION

The main result presented herein is a set of one-dimensional, nonlinear electroelastic equations useful for analyzing wave propagantion and vibrations in quartz bars. These governing equations of successively higher orders of approximation are deduced from the three-dimensional theory of piezoelectricity by a consistent method of reduction. That is, the variational principle (34) together with the series expansions (46) is used to derive the governing equations of quartz bar in which account is taken of only the elastic nonlinearities. The resulting equations incorporate as many higher order effects as deemed necessary in any case of interest by the proper truncation of the series expansions. Thus, the customary use of matching coefficients [6] is eliminated in a rational way. The nonlinear electroelastic equations describe all the higher order stretching, flexure and torsion of thin piezoelectric bars of uniform cross-section. Further, they are easily seen to reduce to the dynamic equations of bars by Mindlin [6,52], Bleustein and Stanley [53], and the author [7,8,51,54], as well as several authors mentioned by them.

The variational principle (33) is obtained from Hamilton's principle by modifying it through Friedrichs's transformation. As its Euler equations, the principle leads to all the fundamental equations of piezoelectricity but the initial conditions. By dropping out the nonlinear elastic terms, the variational principle can be specialized to contain some of earlier variational principles [1,6,25,29,30, 32,33,39,55,56]. The principle permits simultaneous approximation on all the field variables, and hence it is most frequently desirable and compulsory in selecting the trial functions of approximate direct solutions [34,35,45]. Further, special motions are pointed out, the linearized governing equations and the electroelastic equations in the presence of initial stresses are recorded for the quartz bar of uniform cross-section. The uniqueness is examined in solutions of the initial mixed-boundary value problem defined by the linearized governing equations, and the sufficient conditions for the uniqueness are enumerated by means of the logarithmic convexity arguments. It is worth noting that the uniqueness is established even though elasticities neither possess major symmetry (12) nor satisfy a definiteness condition of energies [47, 56].

In closing, the results presented herein can be readily extended to the case in which the thermal effect [57-60], and/or the mechanical effect of the electrode coating [8], are taken into account. Likewise, for a piezoelectric bar with temperature-dependent properties [61], the nonlinear electroelastic equations of higher orders of approximation can be formulated. Further, it is worthwhile to conclude the paper that work [45], is now in progress for certain vibrations of quartz bar, and it will be reported elsewhere.

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CHAPTER 4

VIBRATIONS OF PIEZOELECTRIC DISCS UNDER INITIAL STRESSES

ABSTRACT

A system of two-dimensional equations is derived to govern high frequency motions of piezoelectric discs (plates) under initial stresses. The approximate governing equations are deduced from a three dimensional quasi-variational principle of piezoelectricity by expanding the electric potential and the incremental components of mechanical displacement in a series of Jacobi's polynomials. These equations in an invariant form are applicable to all the types of extensional, flexural and torsional motions of piezoelectric strained discs. Besides, they incorporate as many higher order effects as deemed necessary in any case of interest by a proper truncation of the series. Further, some special cases, and in particular, the case of piezoelectric unstrained discs and the uniqueness for its solutions are indicated.

Key Words: piezoelectricity, quasi-variational principles, plate vibrations, initial stresses, discs.

1- INTRODUCTION

The mathematical modelling of the dynamic response of piezoelectric plates was extensively studied in the literature , and it was reviewed by several authors (e.g.,[1-3]). Most recently, Gerber and Ballato [4] provided almost a complete list of pertinent puplications dealing with dynamic problems of the piezoelectric elements. In view of these, it appears that the application of initial stresses or strains may be utilized to control the performance of certain piezoelectric devices. However, the effect of initial stresses in piezoelectric elements was treated only in a few particular cases. Especially, the propagation of surface acoustic waves was investigated both analytically and experimentally in a piezoelectric continuum with initial stresses [5,6]. In addition, a quasi-variational principle was recently derived to govern the motions of piezoelectric strained continua [7]. Now, an attempt is made to develop consistently the two-

dimensional equations in an invariant form, of successively higher orders of approximation for piezoelectric strained discs of any geometrical shape.

The presence of initial stresses or strains may significantly change the static and dynamic behavior of structures (e.g., [8,9]). Revealing this fact, Thurston [10] studied the wave propagation in stressed crystals under hydrostatic pressure. Herrmann and Armenakas [11] investigated the vibrations and stability of elastic plates under initial stresses. Further, Lee and his colleagues (e.g.,[12,13]) treated the high-frequency vibrations of crystal plates so as to predict changes in the resonant frequencies due to initial stresses. Additional references dealing with the effect of initial stresses in plates were compiled by the author [14]. Moreover, in the absence of initial stresses, one should mention the recent works of Bogy and his students [15,16], Karlash [17], Zaretskii-Feoktistov [18], Baboux and his colleagues [19] and Pan-fu [20] for various problems of piezoelectric discs. As for piezoelectric plates or discs with initial stresses, this is precisely the topic of this paper.

In this paper, the method of reduction due to Mindlin [21] is applied to derive a system of two-dimensional governing equations of piezoelectric plates (discs) under initial stresses. In the first stage, the three-dimensional differential equations of piezoelectric strained continua are expressed by means of a quasi-variational principle [7]. Then, the geometry of a piezoelectric disc is described, certain regularity assumptions are introduced, and the electric potential and the incremental components of disc are expanded in a series of Jacobi's polynomials. Also, the higher orders components of stress, electric displacements and surface loads are defined in consistent with the series expansions. In the next stage, the governing equations of piezoelectric strained discs are consistently and systematically formulated by using the quasi-variational principle together with the series expansions of field quantities. The governing equations incorporate as many higher order effects as deemed desirable, and they take into account for the coupling between extensional, flexural and torsional modes. Lastly, special cases and in particular, the case of piezoelectric unstrained discs are pointed out, and the results are briefly discussed.

NOTATION

In the paper, standard tensor notation is used in a Euclidean 3-space. Accordingly, Einstein's summation convention is implied for all repeated Latin indices (1,2,3) and Greek indices (1,2). Superposed dots are assigned for time differentiations, primes for partial differentiations with respect to the thickness coordinate x^3, and commas and semicolons for partial and covariant differentiations with respect to space coordinates, respectively. Further, a piezoelectric region B with its boundary surface $\partial B(=S_t \cup S_u = S_d \cup S_p)$ is referred to by a fixed, right-handed system of curvilinear coordinates x^k in the space. The symbol B(t) refers to the region B at time t and n_k to the unit outward vector normal to ∂B. Asterisks are used to indicate prescribed quantities. The time interval is denoted by $T=[t_0,t_1)$ and the thickness interval by $H=[-h,h]$.

2- THREE-DIMENSIONAL EQUATIONS OF PIEZOELECTRICITY

The three-dimensional fundamental equations to govern the motions of a piezoelectric strained continuum are summarized in differential form as follows [1,9,5,6].

Divergence Equations:

$$T^{ij}{}_{;i} - \rho b^j = 0 \; ; \quad T^{ij} = \sigma_o^{ik} u^j{}_{;k} + \sigma^{ij} \tag{1}$$

$$D^i{}_{;i} = 0 \tag{2}$$

Gradient Equations

$$S_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i}) \tag{3}$$

$$E_i = -\phi_{,i} \tag{4}$$

Constitutive Equations

$$\sigma^{ij} = C^{ijkl} S_{kl} - C^{kij} E_k \tag{5}$$

$$D^i = C^{ijk} S_{jk} + C^{ij} E_j \tag{6}$$

Boundary Conditions

$$T^j_* - n_i T^{ij} = 0 \qquad \text{on } S_t \tag{7}$$

$$u^*_i - u_i = 0 \qquad \text{on } S_u \tag{8}$$

$$\sigma^* - n_i D^i = 0 \qquad \text{on } S_d \tag{9}$$

$$\phi^* - \phi = 0 \qquad \text{on } S_p \tag{10}$$

Initial Conditions

$$u_i(x^j,t_o) - v_i^{\,*}(x^j) = 0 \quad ,$$

$$\dot{u}_i(x^j,t_o) - w_i^{\,*}(x^j) = 0 \quad ,$$

$$\phi(x^j,t_o) - \phi^*(x^j) = 0 \qquad \text{in } B(t_o) \tag{11}$$

In the above equations, T^{ij} is the stress tensor, u_i the incremental displacement vector, ρ the mass density, $b_j(=\ddot{u}_j)$ the acceleration vector, $\sigma_o{}^{ij}$ and σ^{ij} the initial and incremental stress tensors, D^i the electric displacement vector, E_i the quasi-static electric field vector, ϕ the electric potential, S_{ij} the incremental strain tensor, $T^j(=n_i T^{ij})$ the stress vector and $\sigma(=n_i D^i)$ the surface charge. Also, C^{ijkl}, C^{ijk} and C^{ij} denote the elastic, piezoelectric and dielectric material constants with their usual symmetry properties in the form

$$C^{ijkl} = C^{jikl} = C^{klij},\; C^{ijk} = C^{ikj},\; C^{ij} = C^{ji} \tag{12}$$

3- A QUASI-VARIATIONAL PRINCIPLE

The fundamental differential equations (1)-(11) can be alternatively expressed by means of a quasi-variational principle in the form.

$$\delta I = \delta I^\alpha{}_\alpha + \delta J^\alpha{}_\alpha + \delta L^\alpha{}_\alpha + \delta K^\alpha{}_\alpha + \delta N^\alpha{}_\alpha = 0 \tag{13}$$

with

$$\delta I^1_1 = \int_T dt \,\{ \int_B \left[(\sigma_o{}^{ik} u^j{}_{;k} + \sigma^{ij})_{;i} - \rho b^j \right] \delta u_j \, dV$$
$$+ \int_{S_t} \left[T^j_* - n_i (\sigma_o{}^{ik} u^j{}_{;k} + \sigma^{ij}) \right] \delta u_j \, dS \,\} \tag{14}$$

$$\delta I_2^2 = \int_T dt\{\int_B -D^i{}_{;i}\delta\phi dv + \int_{S_d}(n_i D^i - \sigma^*)\delta\phi dS\} \tag{15}$$

$$\delta J_1^1 = \int_T dt \int_B \left[S_{ij} - \frac{1}{2}(u_{i;j} + u_{j;i})\right]\delta T^{ij} dV \tag{16}$$

$$\delta J_2^2 = \int_T dt \int_B -(E_i + \phi_{,i})\delta D^i dV \tag{17}$$

$$\delta L_1^1 = \int_T dt \int_B \left[\sigma^{ij} - (C^{ijkl}S_{kl} - C^{kij}E_k)\right]\delta S_{ij} dV \tag{18}$$

$$\delta L_2^2 = \int_T dt \int_B \left[D^i - (C^{ijk}S_{jk} + C^{ij}E_j)\right]\delta E_i dV \tag{19}$$

$$\delta K_1^1 = \int_T dt \int_{S_u} (u_i^* - u_i)\delta T^i dS \tag{20}$$

$$\delta K_2^2 = \int_T dt \int_{S_p} (\phi^* - \phi)\delta\sigma dS \tag{21}$$

and

$$\delta N_1^1 = \int_B \rho \{[\dot{u}_i(x^j,t_o) - w_i^*(x^j)]\delta u_i(x^j,t_o) + [u_i(x^j,t_o) - v_i^*(x^j)]\delta u^i(x^j,t_o)\}\, dV \tag{22}$$

$$\delta N_2^2 = \int_B \rho\left[\phi(x^j,t_o) - \psi^*(x^j)\right]\delta\phi(x^j,t_o) dV \tag{23}$$

The quasi-variational principle (12) is fully unconstrained, and it evidently leads to all the fundamental differential equations of piezoelectricity (1)-(12) as the appropriate Euler-Lagrange equations; and conversely, if the fundamental differential equations are met, the quasi-variational principle is clearly satisfied. This principle is recently deduced from Hamilton's principle by the author [7], and it can be similarly obtained from the principle of virtual work as will be reported in a forthcoming communication.

4- GEOMETRY OF A PIEZOELECTRIC DISC

Consider a piezoelectric disc of any geometrical shape, embedded in the Euclidean 3-space. The piezoelectric disc of thickness 2h is referred to the system of curvilinear coordinates x^k, with the faces, of area A, at $x^3=\pm h$ and with x^α the coordinates on the midplane which intersects the

right cylindrical boundary of the disc in Jordan curve C. The disc is coated with perfectly conducting electrodes on both its faces. Further, one should recall the fundamental assumption of the form

$$(2h/d) \ll 1 \tag{24}$$

where d is a characteristic length of disc. This allows one to treat the disc (plate) as a two-dimensional mathematical model of a three-dimensional body.

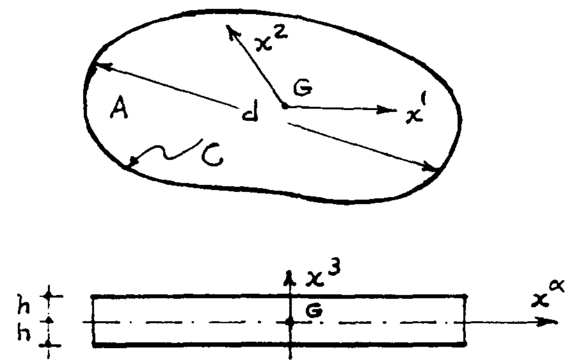

Figure 1. Disc geometry

5- SERIES OF ELECTRIC POTENTIAL AND INCREMENTAL DISPLACEMENTS

The fundamental assumption (24) and the absence of any kind of singularities as well as the suitable regularity and smoothness assumptions are considered for the piezoelectric disc region $B+\partial B$. In addition, all the field functions together with their derivatives are assumed to exist and to be continuous in the closure of disc $\bar{B}(=B\cup\partial B)$, and not to vary widely across the disc thickness. Thus, the electric potential and the incremental components of displacements of disc are approximated by a series in the thickness coordinate as

$$u_i(x^j,t) = \sum_{n=0}^{N=\infty} Q_n^{(i)}(x^3)\, u_i^{(n)}(x^\alpha,t), \tag{25}$$

$$\phi(x^j,t) = \sum_{n=0}^{N=\infty} P_n(x^3)\, \phi_n(x^\alpha,t) \tag{26}$$

Here, the functions Q_n^i and P_n are consistently chosen as Jacobi's polynomials [22,23] in the form

$$Q_n^i = P_n = J_n(x^3) \tag{27}$$

with

$$J_n(z) = 1, z, \ 1-2z^2, \ z-\frac{3}{2}z^3, \ldots, \tag{28}$$

The choice of Eqs. (25)-(27) as a starting point leads to the governing equations of piezoelectric disc in a consistent and tractable manner; this will be shown below. Moreover, in lieu of Jacobi's polynomials, Legendre's polynomials, power series and trigonometric series can be similarly chosen [1-3]. The present choice, however, is more fruitful in the case of circular and elliptical discs.

6- HIGHER ORDER COMPONENTS OF STRESS AND ELECTRIC DISPLACEMENTS AND SURFACE LOADS

In accordance with the foregoing assumptions and the series expansions (25)-(27), the two-dimensional stress and electric displacement components and surface loads of order n in the form

$$\left[\underset{(n)}{T^{\alpha j}}, \underset{(n)}{D^{\alpha}}, \underset{(n)}{N^{j}}, \underset{(n)}{D^{3}}\right] = \int_H \left[(\sigma^{\alpha j}, D^{\alpha}) J_n; (\sigma^{3j}, D^3) J'_n\right] dx^3$$

$$\left[T^{i\alpha}_{o(m+n)}, T^{\alpha 3}_{o(m+n)}\right] = \int_H \left[(\sigma_o^{i\alpha} J_m, \sigma_o^{\alpha 3} J'_m) J_n\right] dx^3$$

$$\left[N^{\alpha}_{o(m+n)}, N^{3}_{o(m+n)}\right] = \int_H \left[(\sigma_o^{3\alpha} J_m, \sigma_o^{33} J'_m) J'_n\right] dx^3$$

$$\left[F^{\alpha}_{o(m+n)}, F^{3}_{o(m+n)}\right] = \left[(\sigma_o^{3\alpha} J_m, \sigma_o^{33} J'_m) J_n\right]\Big|_H$$

$$\left[F^{j}_{(n)}, G_{(n)}\right] = \left[(\sigma^{3j}, D^3) J_n\right]\Big|_H$$

$$\left[\underset{(n)}{T^{j}_{\star}}, \underset{(n)}{D_{\star}}\right] = \int_H \left[(T^{j}_{\star}, \sigma^{\star}) J_n dx^3\right] \tag{29)-(32}$$

and

$$(I_m, I_{mn}) = \int_H (J_m, J_m J_n) dx^3 \tag{33}$$

are defined.

7- HIGHER ORDER, EQUATIONS OF MOTION AND ASSOCIATED NATURAL TRACTION BOUNDARY CONDITIONS

To derive the higher order equations of motion of piezoelectric strained disc, the variational integral (14) is splitted into the area integral over the midplane A and the integral across the thickness, namely,

$$\delta I^1{}_1 = \int_T dt \{ \int_A dA \int_H [(\sigma_o^{ik} u^j{}_{;k} + \sigma^{ij})_{;i} - \rho b^j] \delta u_j dx^3$$

$$+ \oint_C ds \int_H [T_*^j - n_i (\sigma^{ij} + \sigma_o^{ik} u^j{}_{;k})] \delta u_j \, dx^3 \} \qquad (34)$$

Here, the tractions are taken to be prescribed on the edge boundary surface of disc, and accordingly, the displacements are specified on the faces. Following the method of reduction as in [21], the series expansions of incremental displacement components (25)-(27) are inserted into Eq.(34), and it is integrated with respect to the thickness coordinate. Then, using the two-dimensional divergence theorem and regrouping the higher order components of stress and surface loads, one obtains

$$\delta I^1{}_1 = \int_T dt \sum_{n=0}^{N} \{ \int_A \chi_n^j \, \delta u_j^{(n)} dA + \oint_C \chi_n^{*j} \delta u_j^{(n)} dx^3 \} \qquad (35)$$

This equation leads to the equations of motion and the natural boundary conditions, of order n for arbitrary and independent variations $\delta u_i^{(n)}$ in the quasi-variational equation (13). Thus, the equations of motion of order n are expressed by

$$\chi_n^j = T_{n;\alpha}^{\alpha j} - N_n^j + F_n^j + \sum_{m=0}^{N} (T_{o(m+n)}^{\beta\alpha} u^j_{m;\alpha} + T_{o(m+n)}^{\beta 3} u_m^j)_{;\beta}$$

$$- \sum_{m=0}^{N} (N_{o(m+n)}^{\alpha} u^j_{m;\alpha} + N_{o(m+n)}^{3} u_m^j) + \sum_{m=0}^{N} (F_{o(m+n)}^{\alpha} u^j_{m;\alpha}$$

$$+ F_{o(m+n)}^{3} u_m^j) - \rho \sum_{m=0}^{N} I_{mn} \ddot{u}_m^j = 0; \; n=1,2,\ldots,N \quad \text{on} \quad A \qquad (36)$$

Besides, the natural boundary conditions of order n are written in the form

$$\chi_n^{*j} = T_n^{*j} - n_\alpha [T_n^{\alpha j} + \sum_{m=0}^{N} (T_{o(m+n)}^{\alpha\beta} u^j_{(m);\beta} + T_{o(m+n)}^{\alpha 3} \; u^j_{(m)})] = 0$$

$$n=1,2,\ldots,N \text{ along } C \tag{37}$$

8- HIGHER ORDER, EQUATIONS OF ELECTROSTATICS AND ASSOCIATED NATURAL BOUNDARY CONDITIONS OF SURFACE CHARGE

By paralelling to the derivation above, the variational integral (15) is evaluated, the definitions of Eqs.(29)-(32) are used, and then the equations of electrostatics of order n are expressed by

$$Y_n = D^{\alpha}_{(n);\alpha} + G_{(n)} - D^{3}_{(n)} = 0;\ n=1,2,\ldots,N \text{ on } A \tag{38}$$

and the natural boundary conditions of surface charge of order n by

$$Y^{\star}_n = D^{\star}_{(n)} - n_{\alpha} D^{\alpha}_{(n)} = 0;\ n=1,2,\ldots,N \text{ along } C \tag{39}$$

where the surface charges are taken to be prescribed only on the edge boundary of disc.

9- HIGHER ORDER, DISPLACEMENT AND ELECTRIC POTENTIAL BOUNDARY CONDITIONS

The electric potential and the mechanical displacements are considered to be given on the faces. Accordingly, the variational integrals (20) and (21) are carried out on A. Then, the n-th order boundary conditions of displacements are obtained as

$$u_i^{(n)} - u_i^{\star(n)} = 0;\quad n=1,2,\ldots,N \quad \text{on } A \tag{40}$$

and those of electric potential as

$$\phi^{(n)} - \phi_{\star}^{(n)} = 0;\quad n=1,2,\ldots,N \quad \text{on } A \tag{41}$$

10- DISTRIBUTIONS OF INCREMENTAL STRAIN AND ELECTRIC FIELD

By inserting the series expansions (25)-(27) into the variational equations (16) and (17), and then integrating

with respect to the thickness coordinate and using the higher order components of stress and electric displacements, one readily arrives at the distributions of incremental strain and those of electric field in the form

$$(S_{\alpha\beta}, S_{\alpha 3}; E_{\alpha}, E_3) = \sum_{n=0}^{N} (S^{(n)}_{\alpha\beta} J_n,\ S^{(n)}_{\alpha} J'_n + S^{(n)}_{\alpha 1} J_n,\ S^{(n)}_{33} J'_n;\ E^{(n)}_{\alpha} J_n,\ E^{(n)}_{3} J'_n) \qquad (42)$$

where

$$S^{(n)}_{\alpha\beta} = \frac{1}{2}(u^{(n)}_{\alpha;\beta} + u^{(n)}_{\beta;\alpha});\quad S^{(n)}_{\alpha 3} = S^{(n)}_{\alpha} + S^{(n)}_{\alpha 1}$$
$$S^{(n)}_{\alpha} = \frac{1}{2} u^{(n)}_{\alpha},\quad S^{(n)}_{\alpha 1} = \frac{1}{2} u_{3;\alpha};\quad S^{(n)}_{33} = u^{(n)}_{3} \qquad (43)$$

and

$$E^{(n)}_{\alpha} = -\phi^{(n)}_{,\alpha},\quad E^{(n)}_{3} = -\phi^{(n)} \qquad (44)$$

are introduced.

11- MACROSCOPIC CONSTITUTIVE EQUATIONS

With the help of the distributions of electric field and incremental strain (42)-(44), the variational equations (18) and (19) are evaluated and hence the macroscopic constitutive equations of order n are obtained as follows.

$$T^{\alpha j}_{(n)} = \sum_{m=0}^{N} \Big[A^{\alpha j\beta\gamma}_{m+n} S^{(m)}_{\beta\gamma} + 2(B^{\alpha j\beta 3}_{m+n} S^{(m)}_{\beta} + A^{\alpha j\beta 3}_{m+n} S^{(m)}_{\beta 1}) + B^{\alpha j 33}_{m+n} S^{(m)}_{3} - (A^{\beta\alpha j}_{m+n} E^{(m)}_{\beta} + B^{3\alpha j}_{m+n} E^{(m)}_{3})\Big] \qquad (45)$$

$$N^{j}_{(n)} = \sum_{m=0}^{N} \Big[B^{3j\beta\gamma}_{n+m} S^{(m)}_{\beta\gamma} + 2(C^{3j\beta 3}_{m+n} S^{(m)}_{\beta} + B^{3j\beta 3}_{n+m} S^{(m)}_{\beta 1}) + C^{3j33}_{m+n} S^{(m)}_{3} - (B^{\alpha 3j}_{n+m} E^{(m)}_{\alpha} + C^{33j}_{m+n} E^{(m)}_{3})\Big] \qquad (46)$$

$$D^{\alpha}_{(n)} = \sum_{m=0}^{N} \Big[A^{\alpha\beta\gamma}_{m+n} S^{(m)}_{\beta\gamma} + 2(B^{\alpha\beta 3}_{m+n} S^{(m)}_{\beta} + A^{\alpha\beta 3}_{m+n} S^{(m)}_{\beta 1}) + B^{\alpha 33}_{m+n} S^{(m)}_{3} + (A^{\alpha\beta}_{m+n} E^{(m)}_{\beta} + B^{\alpha 3}_{m+n} E^{(m)}_{3})\Big] \qquad (47)$$

$$D^{3}_{(n)} = \sum_{m=0}^{N} \Big[B^{3\beta\gamma}_{n+m}\, S^{(m)}_{\beta\gamma} + 2\,(C^{3\beta 3}_{m+n} S^{(m)}_{\beta} + B^{3\beta 3}_{n+m}\, S^{(m)}_{\beta 1})$$

$$+C^{333}_{m+n}\, S^{(m)}_{3} + (B^{3\beta}_{n+m} E^{(m)}_{\beta} + C^{33}_{m+n} E^{(m)}_{3})\Big] \tag{48}$$

where the higher order components of material constants are defined by

$$(A^{i\ldots j}_{m+n},\ B^{i\ldots j}_{m+n}) = \int_H (C^{i\ldots j} J_m J_n, C^{i\ldots j} \acute{J}_m J_n)\, dx^3$$

$$(C^{i\ldots j}_{m+n}) = \int_H (C^{i\ldots j} \acute{J}_m \acute{J}_n)\, dx^3 \tag{49}$$

12- HIGHER ORDER INITIAL CONDITIONS

As before, the variational equation (23) is evaluated, and the initial conditions of displacements and electric potential of order n are expressed by

$$u^{(n)}_i (x^\alpha, t_o) - v^{*(n)}_i \ (x^\alpha) = 0\ ,$$

$$\dot{u}^{(n)}_i (x^\alpha, t_o) - w^{*(n)}_i (x^\alpha) = 0 \tag{50}$$

$$\phi^{(n)} (x^\alpha, t_o) - \Psi^{*(n)} (x^\alpha) = 0 \tag{51}$$

where $v^{*(n)}_i$, $w^{*(n)}_i$ and $\Psi^{*(n)}$ are the given functions of incremental displacements and electric potentials at time $t=t_o$.

13- GOVERNING EQUATIONS OF PIEZOELECTRIC STRAINED DISCS

The system of two-dimensional equations of piezoelectric discs (plates) under initial stresses consists of the series expansions of incremental strain and electric potential (25)-(27), the higher order equations of motion and elecrostatics (36) and (38), the associated natural boundary conditions (37) and (39)-(41), the distributions of incremental strain and electric field (42)-(44), the macroscopic constitutive equations (45)-(48) and the natural initial conditions (50) and (51).

14- SPECIAL CASES

The approximate, successively higher orders governing equations of piezoelectric strained discs are formulated in an invariant form, and hence they are readily applicable to an arbitrarily shaped disc using a particular coordinate system most suitable for its geometrical configuration. Among those, the resulting equations for a circular discs can be given by using the system of polar coordinates, that is, $x^1=r$, $x^2=\theta$ and $x^3=z$. Likewise, the system of elliptical coordinates can be selected for an elliptical disc under initial stresses.

In the absence of initial stresses, the two-dimensional equations derived may be reduced to those of piezoelectric unstrained discs. These linear governing equations accommodate high frequency motions of a piezoelectric plate (disc), and have a unique solution in each case of interest. The boundary and initial conditions (37) and (39)-(41) which now exclude the terms involving initial stresses and hence become linear are sufficient for the uniqueness. To prove this, utilizing the technique due to Knops and Payne [24] and following the author [25], the existence of two possible solutions is considered and the homogeneous governing equations are formed for the difference of solutions. Then, a logarithmic function is introduced, and it is calculated in terms of the field quantities of disc. By using the convexity of the function and Schwartz's inequality, it is shown that the homogneous problem may possess only a trivial solution. Accordingly, the linearized version of the initial and boundary conditions (37)-(41) are found to be sufficient for the uniqueness [26].

15- CONCLUDING REMARKS

Presented herein is the system of two-dimensional equations of successively higher orders of approximation for all the types of extensional, flexural and torsional motions of piezoelectric discs (plates) under initial stresses. These governing equations are systematically and consistently deduced from the three-dimensional equations of piezoelectricity by means of a quasi-variational principle together with the series expansions of the field quantities. The effects of elastic stiffeness and inertia of electrodes are omitted, but those of shear and normal strains, full

anisotropy and heterogeneity are all taken into account. Then, some special cases, and in particular, the case of unstrained piezoelectric disc and the uniqueness for its solution are pointed out. In closing, the case of circular discs with and without initial stresses is reported in the reference [27], and the detailed analyses of certain motions of strained elliptical discs and the extension of the present results to those of composite discs with initial stresses will be studied in a forthcoming memoir.

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CHAPTER 5

SHELL THEORY FOR VIBRATIONS OF PIEZOCERAMICS UNDER A BIAS

ABSTRACT

This paper is addressed to a consistent derivation of the shell theory in invariant form for the dynamic fields superposed on a static bias of piezoceramics. In the first part of the paper, the fundamental equations of piezoelectric media under a static bias are expressed by the Euler-Lagrange equations of a unified variational principle. The variational principle is deduced from the principle of virtual work by augmenting it through Friedrichs's tranformation. In the second part, a set of two-dimensional, approximate equations of thin elastic piezoceramics is systematically derived by means of the variational principle together with a linear representation of field variables in the thickness coordinate. The two-dimensional electroelastic equations accounting for the influence of mechanical biasing stresses accomodate all the types of incremental motions of a polarized ceramic shell coated with very thin elecrodes. In the third part, emphasis is placed on special motions, geometry and material of piezoceramic shell. Especially, attention is confined to the linearized electroelastic equations of piezoceramic shell, and the uniqueness in their solutions is established by the sufficient boundary and initial conditions.

1- INTRODUCTION

PIEZOCERAMICS are a class of synthetic materials made of anisotropic crystalline powders by pressing, casting, or extrusion, and sintering, and then by prepolarizing under a strong electric field. This poling process induces the piezoelectric properties in ceramics; this is analogous to the magnetizing of magnets or the polarizing of electrets. The piezoceramic materials are chemically as well as physically stable and robust, insensitive to aging and, in particular, potentially low cost. They can be manufactured in a wide range of compositions with desirable properties and a variety of advantages shapes and sizes, as the structural elements of acoustic devices. The characterictics and applications of piezoceramics, including the elastic and piezoelectric constants, are availeble [1]-[5]. Owing to their specific features piezoceramic elements and especially, those elements in the shape of thin shells are quickly replacing

natural piezoelectric elements in recent commercial applications. A review of recent contributions on the dynamic applications of piezoelectric and piezoceramic elements can be found [6], [7].

In acoustic devices, biasing stress or strain and/or electric field is a new design feature, and their introduction may be effectively utilized to control the performance of piezoelements, and to select the most suitable operating conditions, in these devices. The presence of a biasing state induced by external perturbations like thermal, mechanical and electrical fields and even magnetic fields can significantly affect the static and dynamic behavior of structural elements (for instance, beams [8], [9], plates [10]-[12] and shells [13], [14]) and the characteristics of BAW and SAW [15]- [19]. In an initially unbiased solid medium, the linear theory of electroelasticity provides an extremely accurate description of waves and vibrations with small amplitudes. However, in a solid medium with induced external perturbations and/or intrinsic nonlinearities, the linear theory becomes unsatisfactory in describing its motions. This fact was widely recognized, and tackled by many investigators in elasticity (for instance, [20]- [22]) and electroelasticity [23]- [26]. In elasticity, the fundamental differential equations of incremental motions were established and their various applications were exhibited [26]- [28]. These equations make available an invaluable tool in investigating the stability of initial deformations of a solid medium. In electroelasticity, Tiersten [24] derived a properly invariant set of the nonlinear differential equations including thermal effects by means of a systematic application of the fundamental axioms of continuum physics. From these general equations, Baumhauer and Tiersten [29]- [31] obtained the differential electroelastic equations for small dynamic fields superposed on a static biasing state of solid medium, and also, for intrinsically nonlinear fields. Moreover, the fundamental equations of incremental motions were expressed as the Euler-Lagrange equations of variational principles in elasticity [32], [33] and, in piezoelectricity [34]- [36]. A clear and elaborate axposition of the subject was presented by Truesdell and Noll [37], Thurston [26] and Bolotin [27].

To reveal the effect of biasing states on the characteristics of vibrations and waves in elastic media, many investigators considered the effect due to the electrode-induced thermal stresses, mounting and acceleration stresses, as well as stresses resulting from the externally applied forces and pressures [38]-[49] and [6], [7] with a list of

extensive references on the subject. In their pioneering works, Truesdell [38], Toupin and Bernstein [21], Thurston [39]-[41] and Brugger [41] treated some small-amplitude wave propagations in finitely strained elastic materials. Nalamwar and Epstein [42] reported the propagation characteristics of SAW in a strained piezoelectric medium, including the experimental and computed results. Also, the influence of temperature-induced biasing strains [43], of flexural biasing stresses [44], and of biasing electric field [45] was investigated on the propagation characteristics. Sinha, Tanski, Lukaszek and Ballato [18] described some analytical and experimental results on the extensional and flexural stress-induced effects on the propagation of piezoelectric surface waves in crystalline quartz. The author [46] dealt with high-frequency motions of piezoelectric plates under initial stresses, and he and nanI'D. [47] with dynamics of piezoelectric strained rods. Yet an investigation concerning incremental motions of piezoelectric ceramic shell under a bias is unavailable in the current open literature; this is precisely the topic of this paper.

Studies concerning the dynamic analysis of piezoceramic shell were devoted either to solutions of their specific motions or to derivations of their two-dimensional, approximate electroelastic equations. Of the former, the radial, flexural and torsional vibrations as well as the propagations of axisymmetric and non-axisymmetric waves were investigated both analytically and experimentally in spherical and cylindrical thin shells with electroded or unelectroded surfaces, polarized in an axial, radial or circumferential direction and driven electrically or mechanically [48]-[57]. The analytical treatment of radial motions includes a piezoceramic hollow sphere [48], a piezoelectric sphere coated with electrodes on its surface in a compressible fluid [49], and a piezoceramic hollow sphere or cylinder filled with a compressible liquid and immersed within a fluid of infinite extent [50]. The numerical analysis of harmonic vibrations of a piezoceramic shell of revolution, coated with electrodes on its outer and inner surfaces was reported [51]. The propagation of axisymmetric and non-axisymmetric waves was considered in a piezoceramic hollow cylinder with radial and axial polarizations [52]-[54]. The interaction effects of the radiation load and various modes of vibrations of a piezoceramic cylindrical shell were examined for the case when the shell with thickness polarization is partially in contact with an acoustic medium and surrounded by a soft shield [55]. Various types of vibrations of a piezoceramic hollow cylinder were studied by Paul and Venkatesan [56], and Matrosov and Ustinov [57] who cited additional works for

special motions of piezoceramic shells.

Derivations of two-dimensional equations of piezoelectric and piezoceramic shells were reported by a number of investigators [58]-[79]. These governing equations of shells were immensely more tractable than the three-dimensional equations of piezoelectricity, and hence, they are prevalent by computational economy. The fundamental equations of piezoelectricity were reduced to the equations of piezoceramic shells by means of a method of reduction that involves an averaging procedure across the thickness and a set of series expansions for the field variables in terms of the thickness coordinate of shell. Of the methods of reduction [80], the method of symbolic integration [58], the asymptotic method [59] and especially the variational method [60]-[62] were applied together with the power series expansions [63], [64] and the series of Legendre and Jacobi polynomials [64]-[66]. In a noteworthy earlier attempt, Toupin [67] formulated the piezoelectric relations and equations of equilibrium for a polarized elastic spherical shell. Within the context of thin elastic shells [81], a theory was developed for vibrations of piezoelectric ceramic shells of revolution [68], [69], radially and tangentially polarized piezoceramic thin shells [70], [71] and viscoelastic piezoceramic shells, including the effect of temperature [72]. Chau [62], [73] dealt with a theory of piezoelectric and piezoceramic shells and Khoma [61], [66] with that of piezoelectric and thermopiezoelectric shells. Kudryavtsev, Parton and Senik [74], [75] derived a refined theory of piezoelectric ceramic shells that takes into account shear strains, as did Rudnitskii and Shul'ga [76]. By use of Mindlin's variational method [82], [83], the author [60], [63], [65], [77]-[79] established a theory of various types of low and high frequency as well as linear and nonlinear vibrations of piezoelectric and piezoceramic shells and thermopiezoelectric laminae, including the sufficient conditions for the uniqueness in its solutions. Besides, a theory of piezoelectric membranes was obtained as the special case of a shell theory where account was taken of electro-magnetic effect [64]. A survey of various theories and problems of piezoceramic shells, together with an update list of references, was compiled [6], [7].

The objective of this paper is (i) to express the fundamental equations of piezoelectric medium under a mechanical bias by the Euler-Lagrange equations of a unified variational principle, by use of this principle together with a linear representation of field variables; (ii) to establish a two-dimensional theory for the motions of polarized ceramic shells coated with thin electrodes, accounting for the

influence of mechanical bias; and then, (iii) to consider special cases and, in particular, the linearized equations of piezoceramic shells and to examine the uniqueness in their solutions.

In the remainder of this section, the content of the paper is specifically given, and then the notation to be used herein is introduced for convenience. To begin with, the three-dimensional fundamental equations of piezoelectricity with extensions to the effects of mechanical bias are expressed as the Euler-Lagrange equations of a unified variational principle deduced from the principle of virtual work by means of Friedrichs's transformation in Section 2. In the next five sections, the set of two-dimensional approximate equations is systematically derived for a prestressed piezoceramic shell by use of Mindlin's method of reduction. Section 3 contains a description of the geometry of ceramic shell and also the relationships between space and surface tensors needed in the subsequent development. In Section 4, a linear representation is introduced for the mechanical displacements and the electric potential, and then, the resultant field quantities averaged over the thickness of piezoceramic shell are defined. The distributions of mechanical strain and quasi-static electric field are given and the macroscopic constitutive relations, both linear and nonlinear, are formulated for the piezoceramic shell in Section 5. The two-dimensional field equations of incremental motions and the associated boundary conditions are consistently established by use of the unified variational principle together with the linear expansions of field variables, and the initial conditions are recorded at the perturbed state of piezoceramic shell in Section 6. Similarly, the static governing equations of piezoceramic shell are formulated via a variational procedure at the unperturbed state in Section 7. Special cases involving the polarization direction, geometry and motions of piezoceramic shell are indicated, and the governing equations of a biased piezoceramic plate of arbitrary shape and those of an unbiased piezoceramic shell are explicitly stated in Section 8. Also, the fully linearized governing equations of piezoceramic shell are given and the uniqueness of their solutions is investigated. Some conclusions regarding the results obtained are drawn in the last section.

NOTATION

Throughout the paper, standard tensor notation is freely used in a Euclidean three-dimensional space E. Accordingly, Einstein's summation convention is implied over all repeated Latin indices (1,2,3) and Greek indices (1,2) that stand for space and surface tensors, respectively, unless they are put within parantheses. In the space E, the x^i-system is identified with a fixed, right-handed system of general convected (intrinsic) coordinates. All the field quantities are described in Lagrangian formulation, and a quantity in the initial state is designated by a zero index and a prescribed quantity by an asterisk. Further, a comma stands for partial differentiation with respect to the indicated space coordinate, a superposed dot for time differentiation, and a semicolon and a colon for covariant differentiaton with respect to the indicated coordinate, using the space and surface metrics, respectively. Also, the symbol B(t) refers to a regular, finite and bounded region B contained in the space E at time t, $\bar{B}(=B\cup\partial B)$ to the closure of the region B, with its boundary surface ∂B, $\bar{B}XT$ to the domain of definitions for the functions (x^i,t), $T=[t_o,t_1)$ to the time interval, and $H=[-h,h]$ to the interval across the thickness of piezoceramic shell. As for new quantities, they are defined whenever they first appear.

NOMENCLATURE

x^i	a fixed, right-handed system of general convected coordinates
$2h$	thickness of piezoceramic shell
A, C	area of the midsurface of shell, Jordan curve which bounds A
$T^{ij}, \overset{o}{t}{}^{ij}, t^{ij}$	total, initial and incremental stress tensors
$S_{ij}, \overset{o}{s}_{ij}, s_{ij}$	total, initial and incremental strain tensors
ρ	mass density of the undeformed body
$U_i, \overset{o}{u}_i, u_i$	total, initial and incremental displacement vectors
$T^i, \overset{o}{T}{}^i, Ti$	total, initial and incremental stress vectors
n_i	unit outward vector normal to the boundary surface ∂B
D_i	electric displacement vector
E_i	quasi-static electric field vector
σ	surface charge
ϕ	electric potential
$\mathcal{H}$	electric enthalpy

2-VARIATIONAL FORMULATION FOR STRAINED MEDIA

Variational principles, both differential and integral types, are widely appreciated in succintly expressing the fundamental equations of a medium. Besides, these priciples are valuable in systematically deriving lower order field equations and directly providing approximate solutions, and hence they are used for the purpose of this study as well. Primarily, Tiersten and Mindlin [84], Tiersten [85] and EerNisse [86] developed various variational principles in piezoelectricity, as did Vekovishcheva [87] and the author [36], [88]. In addition, Mindlin [89], Nowacki [90] and the author [65], [91]-[93] presented some variational principles in thermopiezoelectricity. However, only little effort was made to formulate variational principles accounting for the effect of biasing stresses [35], [36] in which Hamilton's principle was used as the basis of derivation. In order to render the present work self-contained, it is the purpose of this section to derive a unified variational principle of piezoelectric strained media by taking the principle of virtual work as a starting point. The reader can be referred to [2] for additional background information and to [6], [7] for recent contributions on the subject.

To begin with, referring to a fixed, right-handed system of general convected coordinates x^i in the space E, a regular, finite and bounded region of piezoelectric elastic medium, $B_o+\partial B_o$, with its boundary surface ∂B_o is considered at its initial unperturbed or reference state at time $t=t_o$. At this initial state, the piezoelectric region is subjected to a finite deformation due to static initial stresses, and it is taken to be self-equilibrating. The piezoelectric region acquires its spatial (perturbed or final) state $B+\partial B$ by an additional vibrational or wave motion of small amplitude which is superposed onto the finite static deformation of piezoelectric region $B_o+\partial B_o$ at the time interval $T=[t_o, t_1)$. Now employing the Lagrangian approach, an extended version of the principle of virtual work is stated for the piezoelectric region at its spatial state as an assertation

$$-\delta\Pi+\delta\Upsilon+\delta^{*}W=0 \qquad (5.1a)$$

with the denotations

$$\delta\Pi=\int_B (T^{ij}\delta S_{ij}-D^i\delta E_i)\,dV, \quad \delta\Upsilon=1/2\,\delta\int_B \rho\dot{U}^i\dot{U}_i\,dV \qquad (1b)$$

$$\overset{*}{\delta}W=\int_{\partial B}(T^i_{*}\delta U_i-\sigma_{*}\delta\phi)\,dS$$

where $\delta^* W$ stands for the work done by external mechanical and electrical forces, and δ^* with an asterisk is used to distinguish it from the variation operator δ. Integrating over the time interval T, (1) may be expressed in the form

$$\delta \mathfrak{L}_1 = \int_T dt \int_B -[(t_o^{ij}+t^{ij})\delta S_{ij} - D^i \delta E_i] dV - \int_T dt \int_B \rho a^i \delta u_i dV + \int_T dt \int_{\partial B} [(\tau_{\star o}^{i}+\tau_\star^i)\delta u_i + q_\star \delta\phi] dS = 0 \tag{2}$$

with the definitions

T^{ij}, t_o^{ij}, t^{ij}	total $(=t_o^{ij}+t^{ij})$, initial and incremental stress tensors
S_{ij}, S_{ij}^o, s_{ij}	total $(=s_{ij}^o+s_{ij})$, initial and incremental strain tensors
ρ	mass density of the undeformed body
a^i	Lagrangian acceleration vector $(=\ddot{u}^i)$
U_i, u_i^o, u_i	total $(=u_i^o+u_i)$, initial and incremental displacement vectors
T^i, τ_o^i, τ^i	total $(=\tau_o^i+\tau^i)$, initial and incremental stress vectors
n_i	unit outward vector normal to a surface element of ∂B
D^i	electric displacement vector
E^i	quasi-static electric field vector
σ	surface charge $(=n_i D^i)$
ϕ	electric potential

By inserting the gradient equations by

$$S_{ij} = E_{ij} + 1/2 U^k{}_{;i} U_{k;j}, \quad E_{ij} = 1/2(U_{i;j}+U_{j;i}) \tag{3}$$

$$E_i = -\phi_{,i} \tag{4}$$

into (2), applying the Green-Gauss transformation of integrals for the regular piezoelectric region, carrying out the indicated variations, and then combining terms in the surface and volume integrals, one obtains a two-field variational principle for the piezoelectric strained medium as

$$\delta \mathfrak{L}_2\{u_i, \phi\} = \int_T dt \int_B (L^i \delta u_i + L\delta\phi) dV + \int_T dt \int_{\partial B} (L_\star^i \delta u_i + L_\star \delta\phi) dS = 0 \tag{5}$$

with the divergence equations of incremental motions by

$$L^j = (t^{ij} + t_o^{ik} u^j{}_{;k})_{;i} - \rho a^j = 0 \quad \text{in } \bar{B}XT \qquad (6)$$

$$L = D^i{}_{;i} = 0 \text{ in } \bar{B}XT \qquad (7)$$

and the associated natural boundary conditions by

$$L_*^j = \tau_*^j - n_i (t^{ij} + t_o^{ik} u^j{}_{;k}) = 0 \quad \text{on } \partial BXT \qquad (8)$$

$$L^* = \sigma_* - n_i D^i = 0 \quad \text{on } \partial BXT \qquad (9)$$

as its Euler-Lagrange equations. In deriving (5), the stress equations of equilibrium and the associated boundary conditions at the initial state as

$$L_o^j = [t_o^{ik} (\delta_k^j + u_{o;k}^j)]_{;i} = 0 \quad \text{in } \bar{B}XT \qquad (10a)$$

$$L_o^{*j} = \tau_{o_*}^j - n_i t_o^{ik} (\delta_k^j + u_{o;k}^j) = 0 \quad \text{on } \partial BXT \qquad (10b)$$

are considered, the usual arguments are implied on the increments of field variables [11], [27], and the constraint conditions of the form

$$\delta u_i = 0, \quad \delta\phi = 0 \quad \text{in } B(t_o) \text{ and } B(t_1) \qquad (11)$$

are imposed, and also the variation, differentiation and integration operators are taken to commute with one another and the variations to obey the axiom of conservation of mass.

In order to describe completely the incremental motions of piezoelectric strained medium, the variational principle (5) is supplemented by the gradient equations (3) and (4), and the constitutive relations in the form

$$t^{ij} = \frac{1}{2}\left(\frac{\partial \mathcal{H}}{\partial S_{ij}} + \frac{\partial \mathcal{H}}{\partial S_{ji}}\right), \quad D^i = -\frac{\partial \mathcal{H}}{\partial E_i}) \qquad (12)$$

where $\mathcal{H}(e_{ij}, E_i, t_o^{ij})$ stands for an electric enthalpy function which contains the initial stresses as parameters [28], the boundary conditions as

$$u_i - u_i^* = 0 \quad \text{on } \partial B_u XT \qquad (13)$$

$$\phi - \phi^* = 0 \quad \text{on } \partial B_\phi XT \qquad (14)$$

in addition to (8) and (9), the initial conditions of the form

$$u_i(x^j, t_o) - v_i^*(x^j) = 0, \; \dot{u}_i(x^j, t_o) - w_i^*(x^j) = 0 \quad \text{in } B(t_o) \qquad (15)$$

$$\Phi(x^i,t_o)-\Psi^*(x^i)=0 \quad \text{in } B(t_o) \tag{16}$$

and the constraint conditions (11). These conditions prevent a simple and free (unconstrained) choice of trial functions in direct approximate solutions, and hence the variational principle (5) becomes almost always inconvenient in computation. To remove the constraint conditions, Friedrichs's transformation is implemented [94], and accordingly, a dislocation potential for each constraint is added to (5) so that all the variations can be treated as free [95]. In doing so, the variational integral (5) is expressed in an augmented form (cf., [28]) by

$$\begin{aligned}\delta\mathfrak{L}_3 = & -\int_T dt\int_B(\delta\mathcal{H}+t_o^{ij}u^k_{;i}\delta u_{k;j})dV-\int_T dt\int_B \rho a^i\delta u_i dV \\ & +\int_T dt\int_{\partial B}(\tau^i_*\delta u_i+\sigma_*\delta\phi)dS+\delta\int_T dt\int_B \Delta^\alpha{}_\alpha dV+\delta\int_T dt\int_{\partial B}\Delta^3{}_3 dS \\ & +\delta\int_T dt\int_{\partial B_\phi}\Delta^4{}_4 dS=0\end{aligned} \tag{17}$$

with the Lagrange undetermined multipliers λ^{ij}, λ^i, μ^i, μ and the denotations by

$$\begin{aligned} & \Delta^1{}_1=\lambda^{ij}\left[S_{ij}-\frac{1}{2}(u_{i;j}+u_{j;i})\right], \quad \Delta^2{}_2=\mu^i(E_i+\phi_{,i}) \\ & \Delta^3{}_3=\lambda^i(u_i-u_i^*), \quad \Delta^4{}_4=\mu(\phi-\Phi)\end{aligned} \tag{18}$$

where (10) and a linearized formulation of initial stresses are utilized [28]. As in (2), by performing the indicated variations in (17), using the Green-Gauss transformation of integrals and assembling pertinent terms, the Lagrange multipliers are identified as

$$\lambda^{ij}=t^{ij}, \quad \mu^i=-D^i, \quad \lambda^i=\tau^i, \quad \mu=n_iD^i=\sigma \tag{19}$$

by use of the fundamental lemma of the calculus of variations. Upon substituting (19) into (17) and on bearing in mind the usual admissibility conditions of field variables [36], one concludes a unified variational principle for the incremental motions of piezoelectric strained medium as

$$\delta\mathfrak{L}\{\Lambda_i\} = \delta J^{\alpha i}{}_{\alpha i} + \delta I^{\alpha\beta}{}_{\alpha\beta} = 0 \tag{20a}$$

with the admissible state

$$\Lambda_i=\{u_i,\ S_{ij},\ t^{ij},\ \tau^i;\ \Phi,\ E_i,\ D^i,\ \sigma\} \tag{20b}$$

and the denotations by

$$(\delta J^{11}{}_{11},\ \delta J^{12}{}_{12}, \delta J^{13}{}_{13}) = \int_T dt \int_B (L^i \delta u_i,\ L^{ij}\delta S_{ij}, K_{ij}\delta t^{ij})dV$$

$$(\delta J^{21}{}_{21}, \delta J^{22}{}_{22},\ \delta J^{23}{}_{23}) = \int_T dt \int_B (L\delta\Phi,\ K^i\delta E_i,\ \ M_i\delta D^i)dV \tag{20c}$$

$$(\delta I^{11}{}_{11},\ \delta I^{12}{}_{12}) = \int_T dt(\int_{\partial B_t} L^i_\star \delta u_i dS, \int_{\partial B_u} K^\star_i \delta\tau^i dS)$$

$$(\delta I^{21}{}_{21},\ \delta I^{22}{}_{22}) = \int_T dt(\int_{\partial B_\sigma} L_\star \delta\Phi dS,\ \int_{\partial B_\Phi} K_\star \delta\sigma\ dS) \tag{20d}$$

those by

$$L^{ij} = t^{ij} - \frac{1}{2}\left(\frac{\partial H}{\partial S_{ji}} + \frac{\partial H}{\partial S_{ij}}\right),\ K^{ij} = S_{ij} - \frac{1}{2}(u_{i;j} + u_{j;i}),\ K^\star_j = u_i - u^\star_j$$

$$K^i = -(D^i + \frac{\partial H}{\partial E_i}),\ M_i = -(E_i + \Phi_{,i}),\ K_\star = \Phi^\star - \Phi \tag{20e}$$

and also those defined in (6)-(9). The unified variational principle (20) evidently yields all the fundamental equations of incremental motions of piezoelectric strained media but the initial conditions and the symmetry of stress tensor, as its Euler-Lagrange equations, and conversely, if the fundamental equations are satisfied, the variational principle is definitely verified.

The unified variational principle (20) operates on the incremental, mechanical displacements, strains, stresses and tractions, and the electric potential, quasi-static electric field, electric displacements and surface charge of piezoelectric strained medium. The usual continuity and differentiability conditions for the field variables, the initial conditions (15) and (16), the conditions (11) and the symmetry condition of incremental stress tensor are imposed on the admissible state Λ_i of (20b). The variational principle (20) recovers that deduced from Hamilton's principle in Cartesian coordinates, and it includes certain earlier variational principles as special cases [33]-[36],[82]-[86]. Moreover, the variational principle (20) should be modified for the linearized constitutive relations by use of the electric enthalpy of the form

$$H = \frac{1}{2}C^{ijkl}S_{ij}S_{kl} - \frac{1}{2}C^{ij}E_iE_j - C^{ijk}E_iS_{jk} \tag{21}$$

which implies the dislocation potentials by

$$L^{ij} = t^{ij} - (C^{ijkl}S_{kl} - C_{kij}E_k),\ L^i = D^i - (C^{ijk}S_{jk} + C^{ij}E_j) \tag{22}$$

in lieu of those defined in (20c). Here, C^{ijkl}, C^{ijk} and C^{ij} denote, in this order, the elastic and piezoelectric strain constants and the dielectric permittivity of piezo-

electric medium, with their usual symmetry properties, namely,

$$C^{ijkl}=C^{klij}=C^{jikl},\quad C^{ijk}=C^{jik},\quad C^{ij}=C^{ji} \tag{23}$$

On the other hand, the unified variational principle (20) takes the form

$$\delta\mathfrak{L}_o\{\Lambda_o\}=\int_T dt\int_{B_o}(L_o^j\delta u_j^o+L_o^{ij}\delta S_{ij}^o+K_{ij}^o\delta t_o^{ij})dV+\int_T dt\int_{\partial B_{to}}L_{*o}^j\delta u_j dS$$
$$+\int_T dt\int_{\partial B_{uo}}K_{*i}^o\delta\tau_o^i dS=0 \tag{24a}$$

with the admissible state

$$\Lambda_o=\{u_i^o,\ S_{ij}^o,\ t_o^{ij},\ \tau_o^i\} \tag{24b}$$

the definitions (10) and those by

$$L_o^{ij}=t_o^{ij}-C^{ijkl}S_{kl}^o,\quad K_{ij}^o=S_{ij}^o-\frac{1}{2}(u_{i;j}^o+u_{j;i}^o+u_{o;i}^k+u_{k;j}^o),$$
$$K_{i*}^o=u_i^o-u_{i*}^o \tag{24c}$$

and it leads to the fundamental equations of piezoelectric medium at its initial state.

The differential variational principles (20) and (24) are derived, in a systematic manner, for the spatial and initial states, respectively, These variational principles are quite general, and can be specialized to formulate a number of differential and integral types of variational principles operating on certain fields (cf.,[36]). Among them, noteworthy are a two-field variational principle in the form

$$\delta\mathfrak{L}_4\{t^{ij},D^i\}=\delta J^{13}_{13}+\delta J^{23}_{23}=0 \tag{25a}$$

which operates on the stresses and the electric displacements, and a three-field variational principle as

$$\delta\mathfrak{L}_5\{u_i,t^{ij};\}=\delta J^{11}_{11}+\delta J^{13}_{13}+\delta J^{21}_{21}+\delta I^{11}_{11}+\delta I^{21}_{21}=0 \tag{25b}$$

which operates on the mechanical displacements, the stresses and the electric potential.

3 - GEOMETRY OF THIN PIEZOCERAMIC SHELL

In the three-dimensional Euclidean space E, consider a thin piezoceramic shell which occupies a finite and bounded, regular region of space B+ B with its boundary surface B. The region of piezoceramic shell is bounded by the edge (or lateral) boundary surface S_e and the lower and upper faces, S_{lf} and S_{uf}. The edge boundary surface S_e is taken to be a right cylindrical surface whose generators lie along the normal to the midsurface A of the shell, and it intersects the midsurface A along a closed, smooth and nonintersecting (Jordan) curve C[96]. An outward unit vector normal to S_e is denoted by ν_i and that to $S_f(=S_{lf} \cup S\quad)$ by n_i. In mathematical terms, the region of shell is defined by

$$2h/R_{min} \ll 1 \tag{26}$$

where 2h stands for the uniform thickness of shell and R_{min} for the least principal radius of curvature of the midsurface A. This fundamental assumption allows to treat the shell region as a two-dimensional medium. Besides, it is a sufficient condition in shifting space and surface tensors.

The region of thin piezoceramic shell is referred to the x^i-system of geodesic normal convected coordinates, with $x^3=0$ on the reference surface A. The x^3-axis is chosen positively upward and the x^α-coordinate curves lie on A. The metric tensors of shell space are given by

$$g_{\alpha\beta}=\mu^{\lambda}_{\alpha}\mu^{\nu}_{\beta}a_{\lambda\nu},\quad g^{\alpha\beta}=(\mu^{-1})^{\alpha}_{\lambda}(\mu^{-1})^{\beta}_{\nu}a^{\lambda\nu},\quad g^{3\alpha}=g_{3\alpha}=0,\quad g_{33}=g^{33}=1 \tag{27}$$

with the shifters of the form

$$\mu^{\alpha}_{\beta}=\delta^{\alpha}_{\beta}-x^3 b^{\alpha}_{\beta},\quad \mu^{\alpha}_{\nu}(\mu^{-1})^{\nu}_{\beta}=\delta^{\alpha}_{\beta},\quad \mu(\mu^{-1})^{\alpha}_{\beta}=\delta^{\alpha}_{\beta}-x^3(b^{\alpha}_{\beta}-\delta^{\alpha}_{\beta}b^{\nu}_{\nu}) \tag{28}$$

and the metric tensor of A as

$$a_{\alpha\beta}=g_{\alpha\beta}(x^\sigma,0),\quad a^{\alpha\beta}=g^{\alpha\beta}(x^\sigma,0),\quad a_{\alpha 3}=a^{\alpha 3}=0,\quad a^{33}=a^{33}=1 \tag{29}$$

Here, $a_{\alpha\beta}$ denotes the first fundamental form of the reference surface, $b_{\sigma\beta}$ its second fundamental form and $c_{\alpha\beta}(=b^{\sigma}_{\alpha}b_{\sigma\beta})$ its third fundamental form. By use of the shifters, the components of a vector field, (χ^i, χ_i) and $(\bar{\chi}^i, \bar{\chi}_i)$, which are referred respectively to the base vectors of shell space and those of reference surface are associated with one another as

$$\chi_\alpha = \mu_\alpha^\nu \bar{\chi}_\nu, \quad \chi^\alpha = (\mu^{-1})^\alpha_\nu \bar{\chi}^\nu, \quad \bar{\chi}_\alpha = (\mu^{-1})^\nu_\alpha \chi_\nu, \quad \bar{\chi}^\alpha = \mu^\alpha_\nu \chi^\nu$$
$$\chi^3 = \chi_3 = \bar{\chi}_3 = \bar{\chi}^3 \tag{30}$$

Also, the relationships of the form

$$\chi_{\alpha;\beta} = \mu_\alpha^\nu (\bar{\chi}_{\nu:\beta} - b_{\nu\beta}\bar{\chi}^3), \quad \chi^\alpha_{:\beta} = (\mu^{-1})^\alpha_\nu (\bar{\chi}^\nu_{:\beta} - b^\nu_\beta \bar{\chi}^3), \quad \chi_{\alpha;3} = \mu^\nu_\alpha \bar{\chi}_{\nu,3}$$
$$\chi_{3;\alpha} = \bar{\chi}_{3,\alpha} + b^\nu_\alpha \bar{\chi}_\nu, \quad \chi^\alpha_{;3} = (\mu^{-1})^\alpha_\nu \bar{\chi}^\nu_{,3}, \quad \chi^3_{;\alpha} = \bar{\chi}^3_{,\alpha} + b_{\alpha\beta}\bar{\chi}^\beta \tag{31}$$
$$\chi^3_{;3} = \chi_{3;3} = \chi_{3,3} = \bar{\chi}^3_{,3} = \bar{\chi}_{3,3}$$

are recorded for later use. Here and henceforth, colons are used to designate covariant derivatives with respect to the indicated coordinate by use of surface metrics and semicolons those by use of space metrics.

Further, the elements of volume dV, of surface dS, on S, of area dA on A and of line ds along C are of the forms

$$dV = \sqrt{g}\, dx^1 dx^2 dx^3 = dS dx^3 = \mu dA dx^3, \quad n_\alpha dS = \mu \nu_\alpha ds dx^3 \tag{32}$$

with

$$\mu = |\mu^\alpha_\beta| = (g/a)^{1/2} = 1 - 2x^3 K_m - (x^3)^2 K_g; \quad g = |g_{ij}|, \quad a = |a_{\alpha\beta}| \tag{33}$$
$$K_m = 1/2\, b^\alpha_\alpha, \quad K_g = |b^\alpha_\beta|$$

Here, K_m and K_g are the mean and Gaussian curvatures of the reference surface, respectively. A more elaborate account of preliminaries from the differential geometry of a surface may be found in [81], [97].

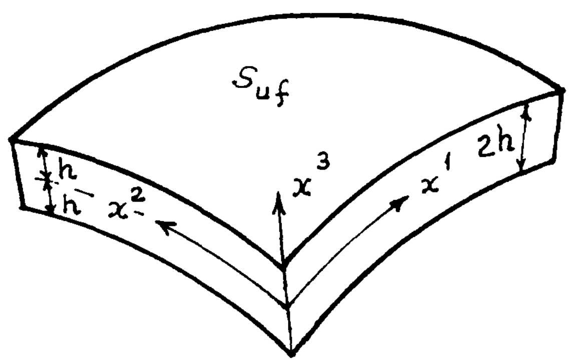

Fig.1. Geometry of piezoceramic shell

4 - MECHANICAL DISPLACEMENTS, ELECTRIC POTENTIALS, AND MECHANICAL AND ELECTRICAL RESULTANTS

All the field variables of thin piezoceramic shell together with their derivatives are taken to be exist and to be single valued and continuous functions of the space coordinates x^i and time t, under suitable regularity and smoothness assumptions for the region of piezoceramic shell $B+\partial B$ with no singularities of any type. Besides, the region of shell is treated as a two-dimensional medium on account of the fundamental assumption (26). In accordance with this, the fields of mechanical displacements and electric potential which are chosen as a starting point of derivation are represented, applying Weierstrass's theorem, by the power series expansions in terms of the thickness coordinate x^3 as

$$\bar{u}_i(x^j,t) = \sum_{n=0}^{N} (x^3)^n u_i^{(n)}(x^\alpha,t) \cong v_i(x^\alpha,t) + x^3 w_i(x^\alpha,t) \qquad (34a)$$

$$\Phi(x^i,t) = \sum_{n=0}^{N} (x^3)^n \, \Phi_{(n)}(x^\alpha,t) \cong \phi(x^\alpha,t) + x^3 \psi(x,t) \qquad (34b)$$

Here, N denotes the order of approximation, and it is taken as N=1, that is, only the zeroth and first order terms are retained in the derivation; this is the closest to the classical theory of thin shells [81]. Also, in (34), $\bar{u}_i$ stands for the components of incremental mechanical displacements referred to the base vectors of the reference surface A defined in (30). The components v_α characterize the extensional motions, v_3 and w_α the flexural motions and w_3 the thickness motions of piezoceramic strained shell.

The representation (34) evidently implies a distribution of incremental strain and quasi-static electric field in the form

$$\{S_{ij}, E_i\} = \sum_{m=0}^{2} (x^3)^m \, \{{}_mS_{ij}, {}_mE_i\} \qquad (35)$$

Here, the incremental strain of order (m), ${}_mS_{ij}$ and the electric field of order (m), ${}_mE_i$ are functions of the aerial coordinates x^α and time t, only. The explicit expressions of electric potential and incremental strain of order (m) are obtained in the next section.

In consistent with the linear representation (34), the electrical and mechanical field quantities are taken not to vary widely across the thickness of piezoceramic shell, and hence they are averaged over the thickness interval $H(=[-h,h])$.

Accordingly, the two-dimensional resultants of incremental stress are defined by

$$[N^{\alpha\beta},\ M^{\alpha\beta},\ K^{\alpha\beta}]=\int_H [1,\ x^3,(x^3)^2]\,t^{\alpha\beta}\mu dx^3$$
$$[Q^\alpha,\ R^\alpha]=\int_H [1,x^3]\,t^{\alpha 3}\mu dx^3,\ N^{33}=\int_H t^{33}\mu dx^3 \tag{36}$$

those of acceleration by

$$A^i=\mu_o\ddot{v}^i-\mu_1\ddot{w}^i,\ B^i=\mu_1\ddot{v}^i-\mu_2\ddot{w}^i \tag{37}$$

with

$$\mu_n=\int_H (x^3)^n\mu dx^3=I_n-2K_m I_{n+1}+K_g I_{n+2} \tag{38}$$

and

$$I_n=\int_H (x^3)^n dx^3;\ I_{2p}=2(h)^{2p-1}/(2p-1),\ I_{2p+1}=0 \tag{39}$$

those of traction by

$$(q^\alpha,\ p^\alpha)=(\mu\mu^\alpha_\beta t^{3\beta}),\ \ (q^3,p^3)=(\mu t^{33})\,\text{at}\,(x^3=h,-h) \tag{40a}$$

and

$$(r^\alpha_o,s^\alpha_o)=\{\mu t_o^{3\beta}[v^\alpha{}_{:\beta}-b^\alpha_\beta v_3+x^3(w^\alpha{}_{:\beta}-b^\alpha_\beta w_3)]+\mu t_o^{33}w^\alpha\}\,\text{at}\,(x^3=h,-h)$$
$$(r^3_o,s^3_o)=\{\mu t_o^{3\alpha}[v_{3,\alpha}+b^\beta_\alpha v_\beta+x^3(w_{3,\alpha}+b^\beta_\alpha w_\beta)]+\mu t_o^{33}w_3\}\,\text{at}\,(x^3=h,-h) \tag{40b}$$

those of loads by

$$(N^\alpha_*,M^\alpha_*)=\int_H \tau^\beta_*\mu^\alpha_\beta(1,x^3)\mu dx^3,\ \ (N^3_*,M^3_*)=\int_H \tau^3_*(1,x^3)\mu dx^3$$
$$(S^\alpha_*,P^\alpha_*)=(\mu\mu^\alpha_\beta\tau^\beta_*)\ \text{and}\ (S^3_*,P^3_*)=(\mu\tau^3_*)\,\text{at}\,(x^3=h,-h \tag{41}$$

and

$$l^i=q^i-p^i,\ l^i_o=r^i_o-s^i_o;\ m^i=h(q^i+p^i),m^i_o=h(r^i_o+s^i_o) \tag{42}$$

those of initial stress by

$$[N_o^{\alpha\beta}\ ,M_o^{\alpha\beta}\ ,\ K_o^{\alpha\beta}\ ]=\int_H [1,x^3,(x^3)^2]\,t_o^{\alpha\beta}\ \mu dx^3$$
$$[Q^\alpha_o,\ R^\alpha_o]=\int_H [1,x^3]\,t_o^{\alpha 3}\mu dx^3 \tag{43}$$

those of initial traction by

$$(q^{\alpha}_{o},\ p^{\alpha}_{o}) = \mu t^{3\beta}_{o}\left[\mu^{\alpha}_{\beta} + v^{\alpha}_{o:\beta} - b^{\alpha}_{\beta} v^{o}_{3} + x^{3}(w^{\alpha}_{o;\beta} - b^{\alpha}_{\beta} w^{o}_{3})\right] + \mu t^{33}_{o} w^{\alpha}_{o} \quad \text{at}\,(x^{3}=h,-h)$$

$$(q^{3}_{o},\ p^{3}_{o}) = \mu t^{3\alpha}_{o}\left[v^{o}_{3,\alpha} + b^{\alpha}_{\beta} v^{o}_{\beta} + x^{3}(w^{o}_{3,\alpha} + b^{\beta}_{\alpha} w^{o}_{\beta}) + \mu t^{33}_{o}(1 + w^{o}_{3})\right] \tag{44}$$

and those of initial loads by

$$(N^{\alpha}_{*o},\ M^{\alpha}_{*o}) = \int_H \tau^{\beta}_{*o} \mu^{\alpha}_{\beta} (1, x^{3}) \mu dx^{3},\ (N^{3}_{*o}, M^{3}_{*o}) = \int_H \tau^{3}_{*o} (1, x^{3}) \mu dx^{3}$$

$$(S^{\alpha}_{*o},\ P^{\alpha}_{*o}) = (\mu \mu^{\alpha}_{\beta} \tau^{\beta}_{*o}) \text{ and } (S^{3}_{*o}, P^{3}_{*o}) = (\mu \tau^{3}_{*o}) \text{ at } (x^{3}=h,-h) \tag{45}$$

and

$$N^{i}_{o} = q^{i}_{o} - p^{i}_{o},\ M^{i}_{o} = h(q^{i}_{o} + p^{i}_{o}) \tag{46}$$

and also, the two-dimensional gross electric displacements by

$$(F^{i},\ G^{i}) = \int_H (1,\ x^{3}) D^{i} \mu dx^{3} \tag{47}$$

surface charge resultants by

$$(c,\ d) = (\mu D^{3}) \text{ at } (x^{3}=h,\ -h),\ e = (c+d)h \tag{48}$$

and edge-surface charge resultants by

$$(F_{*},\ G_{*}) = \int_H \sigma_{*} (1,\ x^{3}) \mu dx^{3} \tag{49}$$

are introduced. In the foregoing definitions, the resultants of stress and the gross electric displacements are measured per unit length of the coordinate curves on the reference surface A, the resultants of acceleration, surface load and surface charge per unit area of A, and those of edge-load and edge-surface charge per unit length of the Jordan curve C of A.

5- STRAIN AND ELECTRIC FIELD DISTRIBUTIONS, CONSTITUTIVE RELATIONS

The distributions of incremental strain and electric field for the piezoceramic shell are expressed in terms of the displacement and electric potential gradients, respectively. To obtain the explicit forms of the distributions of order (m), (35) is inserted into the third term of variational volume integrals in (20), namely,

$$\delta J^{13}_{13}=\int_T dt\int_A dA\int_H K_{ij}\,\delta t^{ij}\mu dx^3=0 \tag{50}$$

and carrying out the integration with respect to the thickness coordinate, one obtains

$$\begin{aligned}\delta J^{13}_{13}=\int_T dt\int_A\Big[&({}_0S_{\alpha\beta}-e_{\alpha\beta})\,\delta N^{\alpha\beta}+({}_1S_{\alpha\beta}-\varepsilon_{\alpha\beta})\,\delta M^{\alpha\beta}+({}_2S_{\alpha\beta}-\gamma_{\alpha\beta})\,\delta K^{\alpha\beta}\\&+({}_0S_{\alpha 3}-e_{\alpha 3})\,\delta Q^{\alpha}+({}_1S_{\alpha 3}-\varepsilon_{\alpha 3})\,\delta R^{\alpha}+({}_0S_{33}-e_{33})\,\delta N^{33}\Big]dA=0\end{aligned} \tag{51}$$

This yields the distributions of incremental strain as

$${}_0S_{ij}=e_{ij}(x^{\alpha},t),\quad {}_1S_{ij}=\varepsilon_{ij}(x^{\alpha},t),\quad {}_2S_{ij}=\gamma_{ij}(x\ ,t) \tag{52a}$$

with the definitions

$$\begin{aligned}&e_{\alpha\beta}=\frac{1}{2}(v_{\alpha:\beta}+v_{\beta:\alpha}-2b_{\alpha\beta}v_3),\quad e_{\alpha 3}=\frac{1}{2}(v_{3,\alpha}+b^{\beta}_{\alpha}v_{\beta}+w_{\alpha}),\quad e_{33}=w_3\\&\varepsilon_{\alpha\beta}=\frac{1}{2}(-b^{\sigma}_{\alpha}v_{\sigma:\beta}-b^{\sigma}_{\beta}v_{\sigma:\alpha}+2c_{\alpha\beta}v_3+w_{\alpha:\beta}+w_{\beta:\alpha}-2b_{\alpha\beta}w_3),\quad \varepsilon_{\alpha 3}=\frac{1}{2}w_{3,\alpha}\\&\gamma_{\alpha\beta}=\frac{1}{2}(-b^{\sigma}_{\alpha}w_{\sigma:\beta}-b^{\sigma}_{\beta}w_{\sigma:\alpha}+2c_{\alpha\beta}w_3);\ \varepsilon_{33}=\gamma_{\alpha i}=0\end{aligned} \tag{52b}$$

as its Euler-Lagrange equations. Likewise, substituting (35) into the variational volume integral ∂J^{23}_{23} of (20) in the form

$$\delta J^{23}_{23}=\int_T dt\int_A dA\int_H M_i\,\delta D^i\mu dx^3=0 \tag{53}$$

and performing the indicated variations, one reads

$$\delta J^{23}_{23}=\int_T dt\int_A\Big[({}_0E_{\alpha}-e_{\alpha})\,\delta F^{\alpha}+({}_1E_{\alpha}-\varepsilon_{\alpha})\,\delta G^{\alpha}+({}_0E_3-e_3)\,\delta F^3\Big]dA=0 \tag{54}$$

which has the distributions of electric field by

$${}_0E_i=e_i(x^{\alpha},t),\quad {}_1E_i=\varepsilon_i(x^{\alpha},t) \tag{55a}$$

with

$$e_\alpha=-\phi_{,\alpha},\quad e_3=-\psi,\qquad \varepsilon_\alpha=-\psi_{,\alpha},\quad \varepsilon_3=0 \tag{55b}$$

In evaluation of (50) and (53), the mechanical and electrical resultants (36) and (47) and the relations (31) are considered.

The distributions of strain and electric field (52) and (55) are now substituted into the constitutive parts of the variational principle (20), and then the variations with respect to the thickness coordinate are carried out recalling the resultants (36) and (47). Thus, the mechanical constitutive part of (20) is expressed by

$$\begin{aligned}\delta J^{12}_{12}=\int_T dt\int_A\{&[N^{\alpha\beta}-\frac{1}{2}(\frac{\partial\Omega}{\partial e_{\alpha\beta}}+\frac{\partial\Omega}{\partial e_{\beta\alpha}})]\delta e_{\alpha\beta}+[M^{\alpha\beta}-\frac{1}{2}(\frac{\partial\Omega}{\partial\varepsilon_{\alpha\beta}}+\frac{\partial\Omega}{\partial\varepsilon_{\beta\alpha}})]\delta\varepsilon_{\alpha\beta}\\&+[K^{\alpha\beta}-\frac{1}{2}(\frac{\partial\Omega}{\partial\gamma_{\alpha\beta}}+\frac{\partial\Omega}{\partial\gamma_{\beta\alpha}})]\delta\gamma_{\alpha\beta}+[Q^{\alpha}-\frac{1}{2}(\frac{\partial\Omega}{\partial e_{\alpha3}}+\frac{\partial\Omega}{\partial e_{3\alpha}})]\delta e_{\alpha3}\\&+[R^{\alpha}-\frac{1}{2}(\frac{\partial\Omega}{\partial\varepsilon_{\alpha3}}+\frac{\partial\Omega}{\partial\varepsilon_{3\alpha}})]\delta\varepsilon_{\alpha3}+(N^{33}+\frac{\partial\Omega}{\partial e_{33}})\delta e_{33}\}dA=0\end{aligned} \tag{56a}$$

and the electrical constitutive part of (20) by

$$\delta J^{22}_{22}=\int_T dt\int_A[(F^i+\frac{\partial\Omega}{\partial e_i})\delta e_i+(G^\alpha+\frac{\partial\Omega}{\partial\varepsilon_\alpha})\delta\varepsilon_\alpha]dA=0 \tag{56b}$$

The Euler-Lagrange equations of (56) and (57) are the constitutive equations of piezoceramic shell in the form

$$\begin{aligned}&N^{\alpha\beta}=\frac{1}{2}(\frac{\partial\Omega}{\partial e_{\alpha\beta}}+\frac{\partial\Omega}{\partial e_{\beta\alpha}}),\quad M^{\alpha\beta}=\frac{1}{2}(\frac{\partial\Omega}{\partial\varepsilon_{\alpha\beta}}+\frac{\partial\Omega}{\partial\varepsilon_{\beta\alpha}}),\quad K^{\alpha\beta}=\frac{1}{2}(\frac{\partial\Omega}{\partial\gamma_{\alpha\beta}}+\frac{\partial\Omega}{\partial\gamma_{\beta\alpha}})\\&Q^{\alpha}=\frac{1}{2}(\frac{\partial\Omega}{\partial e_{\alpha3}}+\frac{\partial\Omega}{\partial e_{3\alpha}}),\quad R^{\alpha}=\frac{1}{2}(\frac{\partial\Omega}{\partial\varepsilon_{\alpha3}}+\frac{\partial\Omega}{\partial\varepsilon_{3\alpha}}),\quad N^{33}=\frac{\partial\Omega}{\partial e_{33}}\quad \text{on } AXT\end{aligned} \tag{57}$$

and

$$F^i=-\frac{\partial\Omega}{\partial e_i},\quad G^\alpha=-\frac{\partial\Omega}{\partial\varepsilon_\alpha}\qquad \text{on } AXT \tag{58}$$

where

$$\Omega=\int_H \mathcal{H}\mu\, dx^3 \tag{59}$$

is the electric enthalpy per unit area of A.

The constitutive equations of piezoceramic shell (57) and (58) include the effect of nonhomogeneity as well as that of nonlinearity. If the effects are neglected, that is, the quadratic version of electric enthalpy (21) is invoked and accordingly (22) is considered in (20), the linear constitutive equations of piezoceramic shell are obtained as

$$(N^{\alpha\beta},\ M^{\alpha\beta},\ K^{\alpha\beta}) = (\underset{\sim}{C}_o, \underset{\sim}{C}_1, \underset{\sim}{C}_2)^{\alpha\beta k}(\underset{\sim}{S}_{\kappa l})^T - (\underset{\sim}{C}_o, \underset{\sim}{C}_1, \underset{\sim}{C}_2)^{k\alpha\beta}(\underset{\sim}{E}_k)^T$$

$$(Q^{\alpha},\ R^{\alpha}) = (\underset{\sim}{C}_o, \underset{\sim}{C}_1)^{\alpha 3kl}(\underset{\sim}{S}_{\kappa l})^T - (\underset{\sim}{C}_o, \underset{\sim}{C}_1)^{k\alpha 3}(\underset{\sim}{E}_k)^T \quad \text{on AXT} \tag{60}$$

$$N^{33} = \underset{\sim}{C}_o^{33kl}(\underset{\sim}{S}_{kl})^T - \underset{\sim}{C}_o^{k33}(\underset{\sim}{E}_k)^T$$

and

$$(F^i,\ G^i) = (\underset{\sim}{C}_o,\ \underset{\sim}{C}_1)^{ijk}(\underset{\sim}{S}_{jk})^T - (\underset{\sim}{C}_o,\ \underset{\sim}{C}_1)^{ik}(\underset{\sim}{E}_k)^T \quad \text{on AXT} \tag{61}$$

Here, the matrices of mechanical strain and electric field are introduced by

$$(\underset{\sim}{S}_{kl}) = (e_{kl},\ \varepsilon_{kl},\ \gamma_{kl}),\quad (\underset{\sim}{E}_k) = (e_k,\ \varepsilon_k,\ 0) \tag{62}$$

and the elastic stiffenesses by

$$\underset{\sim}{C}_n^{ij\ldots k} = (C_n,\ C_{n-1},\ C_{n-2})^{ij\ldots k},\quad C_n^{ij\ldots k} = \int_H C^{ij\ldots k}(x^3)^n \mu dx^3 \tag{63a}$$

which can be expressed in the form

$$C_n^{ij\ldots k} = C^{ij\ldots k}\mu_n \tag{63b}$$

in the case of homogeneous ceramic material.

6 - ELECTROELASTIC EQUATIONS OF INCREMENTAL MOTION, AND ASSOCIATED BOUNDARY AND INITIAL CONDITIONS

This section is devoted to a consistent derivation of the two-dimensional electroelastic equations of incremental motion and the associated boundary and initial conditions for the piezoceramic shell from the three-dimentional equations of piezoelectricity. The point of departure for the systematic derivation is the linearized representation (34) and the unified variational principle (20). The derived equations involve the stress and couple resultants introduced in Section 4. To begin with, the first term of volume integrals $\delta J^{11}{}_{11}$ in (20) is stated as

$$\delta J^{11}{}_{11} = \int_T dt \int_A dA \int_H \left[L^{\alpha}\mu_{\alpha}^{\nu}(\delta v_{\nu} + x^3\delta w_{\nu}) + L^3(\delta v_3 + x^3\delta w_3)\right]\mu dx^3 = 0 \tag{64}$$

where (30) and (34) are used. The integration of this equation with respect to x^3 yields

$$\delta J^{11}{}_{11} = \int_T dt \int_A \left[(V^i + U_o^i + l_i + l_o^i - \rho A^i)\delta v_i + (W^i + T_o^i + m^i + m_o^i - \rho B^i)\delta w_i\right] dA = 0 \tag{65}$$

Here, the mechanical resultants (36)-(46) are recalled, and also, various relations between space and surface tensors and their derivatives are considered and the identities of the form

$$\mu\mu_{\alpha}^{\nu}\chi^{\beta\alpha}{}_{;\beta}=(\mu\mu_{\alpha}^{\nu}\chi^{\beta\alpha})_{:\beta}-\mu\mu_{\alpha}^{\nu}(\mu^{-1})_{\lambda}^{\sigma}\,b_{\sigma}^{\lambda}\chi^{3\alpha}-\mu b_{\alpha}^{\nu}\chi^{\alpha 3}$$
$$\mu\chi^{\alpha 3}{}_{;\alpha}=(\mu\chi^{\alpha 3})_{:\alpha}+\mu\mu_{\alpha}^{\nu}b_{\nu\beta}\chi^{\beta\alpha}-\mu(\mu^{-1})_{\beta}^{\alpha}b_{\alpha}^{\beta}\chi^{33} \tag{66}$$
$$\mu_{\alpha}^{\nu}\chi^{3\alpha}{}_{;3}=(\mu_{\alpha}^{\nu}\chi^{3\alpha})_{,3}$$

and

$$\mu_{,3}=-2(K_m-x^3K_g)=-\mu(\mu^{-1})_{\beta}^{\alpha}b_{\alpha}^{\beta} \tag{67}$$

are used. From (65), it follows at once the two-dimensional equations of incremental motion as

$$V^i+U_o^i+l^i+l_o^i-\rho A^i=0 \quad \text{on AXT} \tag{68}$$
$$W^i+T_o^i+m^i+m_o^i-\rho B^i=0 \quad \text{on AXT}$$

In these equations, the field quantities

$$V^{\alpha}=V^{\beta\alpha}{}_{:\beta}-b_{\nu}^{\alpha}Q^{\nu},\quad V^3=V^{\alpha 3}{}_{:\alpha}+b_{\alpha\beta}V^{\alpha\beta}$$
$$W^{\alpha}=W^{\beta\alpha}{}_{:\beta}-Q^{\alpha},\quad W^3=W^{\alpha 3}{}_{:\alpha}-N^{33}+b_{\alpha\beta}W^{\alpha\beta} \tag{69a}$$

and

$$U_o^{\alpha}=U_o^{\beta\alpha}{}_{:\beta}-b_{\nu}^{\alpha}U_o^{\nu 3},\quad U_o^3=U_{o:\alpha}^{\alpha 3}+b_{\alpha\beta}U_o^{\alpha\beta}$$
$$T_o^{\alpha}=T_o^{\beta\alpha}{}_{:\alpha}-b_{\nu}^{\alpha}T_o^{\nu 3}-Q_o^{\nu}(v^{\alpha}{}_{:\nu}-b_{\nu}^{\alpha}v_3)+R_{o:\beta}^{\beta}-N^{33})w^{\alpha} \tag{69b}$$
$$T_o^3=T_o^{\alpha 3}{}_{:\alpha}+b_{\alpha\beta}T_o^{\alpha\beta}-(R_{o:\alpha}^{\alpha}+N_o^{33})w^3-Q_o^{\alpha}(v_{3,\alpha}+b_{\alpha}^{\beta}w_{\beta})$$

where

$$V^{\alpha\beta}=N^{\alpha\beta}-b_{\nu}^{\beta}M^{\alpha\nu},\quad W^{\alpha\beta}=M^{\alpha\beta}-b_{\nu}^{\beta}K^{\alpha\nu};\quad V^{\alpha 3}=Q^{\alpha},\quad W^{\alpha 3}=R^{\alpha} \tag{69c}$$

and

$$U_o^{\alpha\beta}=N_o^{\alpha\nu}(v^{\beta}{}_{:\nu}-b_{\nu}^{\beta}v_3)+M_o^{\alpha\nu}(w^{\beta}{}_{:\nu}-b_{\nu}^{\beta}w_3)+Q_o^{\alpha 3}w^{\beta}$$
$$U_o^{\alpha 3}=Q_o^{\alpha 3}w_3+N_o^{\alpha\beta}(v_{3,\beta}+b_{\beta}^{\nu}v_{\nu})+M_o^{\alpha\beta}(w_{3,\beta}+b_{\beta}^{\nu}w_{\nu})$$
$$T_o^{\alpha\beta}=M_o^{\alpha\nu}(v^{\beta}{}_{:\nu}-b_{\nu}^{\beta}v_3)+K_o^{\alpha\nu}(w^{\beta}{}_{:\nu}-b_{\nu}^{\beta}w_3) \tag{69d}$$
$$T_o^{\alpha 3}=M_o^{\alpha\beta}(v_{3,\beta}+b_{\beta}^{\nu}v_{\nu})+K_o^{\alpha\beta}(w_{3,\beta}+b_{\beta}^{\nu}w_{\nu})$$

are introduced.

In a similar manner, inserting (34) into the fourth term of volume integrals $\delta J^{21}{}_{21}$ in (20) one reads

$$\delta J^{21}{}_{21}=\int_T dt\int_A dA\int_H -D^i{}_{;i}(\delta\phi+x^3\delta\psi)\mu dx^3=0 \tag{70}$$

Then integrating with respect to x^3 and using the identity (67) and that of the form

$$\mu\chi^\alpha{}_{;\alpha}=(\mu\chi^\alpha)_{:\alpha}+\mu_{,3}\chi^3 \tag{71}$$

the variational integral (70) becomes

$$\delta J^{21}{}_{21}=\int_T dt\int_A\left[(F^\alpha{}_{:\alpha}+c-d)\delta\phi+(G^\alpha{}_{:\alpha}-F^3+e)\delta\psi\right]dA=0 \tag{72}$$

in terms of the resultants (47) and (48). The Euler-Lagrange equations of (72) are given by

$$\left.\begin{array}{l}F^\alpha{}_{:\alpha}+c-d=0\\ G^\alpha{}_{:\alpha}-F^3+e=0\end{array}\right\}\quad \text{on } AXT \tag{73}$$

which denote the two-dimensional charge equations of electrostatics.

Now, attention is turned to the associated boundary conditions which follow easily by evaluating the surface integrals of (20). The mechanical displacements are taken to be prescribed on only a part $S_u(=C_uXH)$ of the edge boundary surface S_e and the tractions on the remaining part $S_t(=C_tXT)$ of S_e and the lower and upper faces S_{lf} and S_{uf}. An alternating potential difference is applied to the perfectly conducting thin electrodes on S_f and the surface charges are specified on $S_e(=CXH)$. Thus, the mechanical surface integrals of (20) are stated in the form

$$\delta I^{11}{}_{11}=\int_T dt\int_{C_t}\int_H\left[\tau^j_*-\nu_\alpha(t^{\alpha j}+t_o^{\alpha k}u^j{}_{;k})\right]\delta u_j\mu ds dx^3$$
$$+\int_T dt\int_{S_f}\left[\tau^j_*-n_3(t^{3j}+t_o^{3k}u^j{}_{;k})\right]\delta u_j\mu dA=0 \tag{74a}$$

and

$$\delta I^{12}{}_{12}=\int_T dt\int_{C_u}\int_H(u_i-u_i^*)\delta\tau^i\mu ds dx^3=0 \tag{74b}$$

Evaluating these integrals as in (64), the natural boundary conditions of mechanical displacements are obtained as

$$v_i - v_i^* = 0, \; w_i - w_i^* = 0 \quad \text{along } C_u XT \tag{75}$$

and those of tractions as

$$N_*^j - \nu_\alpha (V^{\alpha j} + U_o^{\alpha j}) = 0, \; M^j = \nu_\alpha (W^{\alpha j} + T_o^{\alpha j}) \quad \text{along } C_t XT \tag{76}$$

and

$$S_*^j - (q^j + r_o^j) = 0 \text{ on } S_{uf} XT, \; P_*^j - (p^j + s_o^j) = 0 \text{ on } S_{lf} XT \tag{77}$$

in terms of the resultants (40) and (41).

Similarly, the electrical surface integrals of (20) are expressed by

$$\delta I_{21}^{21} = \int_T dt \int_{S_f} (\phi - \phi_*) \delta\sigma\mu dA = 0, \; \delta I_{22}^{22} = \int_T dt \int_C \oint_H (\nu_\alpha D^\alpha - \sigma_*) \delta\phi\mu ds dx^3 = 0 \tag{78}$$

After evaluation, this leads to the natural boundary conditions of electric potential as

$$\phi - \gamma_* = 0, \; \psi - \delta_* = 0 \text{ on } AXT \tag{79a}$$

and

$$\nu_\alpha F_*^\alpha - F = 0, \quad \nu_\alpha G^\alpha - G_* = 0 \quad \text{along } CXT \tag{79b}$$

in terms of the resultants (49).

On the other hand, a set of initial conditions arises in conjuction with (15), (16) and (34) as

$$\left.\begin{array}{l} v_i(x^\beta, t_o) - \alpha_i^*(x^\beta) = 0, \; w_i(x^\alpha, t_o) - \beta_i^*(x^\alpha) = 0 \\ \dot{v}_i(x^\beta, t_o) - \gamma_i^*(x^\beta) = 0, \; \dot{w}_i(x^\alpha, t_o) - \delta^*_i(x^\alpha) = 0 \end{array}\right\} \text{ on } A(t_o) \tag{80}$$

and

$$\phi(x^\alpha, t_o) - \alpha_*(x^\alpha) = 0, \quad \psi(x^\alpha, t_o) - \beta_*(x^\alpha) = 0 \text{ on } A(t_o) \tag{81}$$

where α_i^*, β_i^*, γ_i^*, δ_i^*, α^* and β are the specified functions of the coordinates x^α.

7 - THEORY OF PIEZOCERAMIC SHELL UNDER A BIAS

Thus far, the electroelastic equations of a piezoceramic shell are derived for its incremental motions superposed upon an initial, static finite deformation at its perturbed state. Now, the two-dimensional equations are complemented by those of piezoceramic shell at its unperturbed state. At the unperturbed state, the piezoceramic shell is taken to be self-equilibrating, no electric field to be present and the field of mechanical displacements to be presented by

$$\bar{u}^{\circ}_{i}(x^{j}) = v^{\circ}_{i}(x^{\alpha}) + x^{3} w^{\circ}_{i}(x^{\alpha}) \tag{82}$$

which is the counterpart of (34a) for the static deformation of shell.

In parallel with the derivation of the electroelastic equations at the perturbed state, the variational principle (24) is evaluated, and then the two-dimensional equations of equilibrium are obtained in the form

$$\left.\begin{aligned}
&(\underset{\circ}{V}^{\beta\alpha} + \underset{\circ}{A}^{\beta\alpha})_{:\beta} - b^{\alpha}_{\beta}(\underset{\circ}{Q}^{\beta} + \underset{\circ}{A}^{\beta}) + \underset{\circ}{N}^{\beta} = 0 \\
&(\underset{\circ}{V}^{\alpha 3} + \underset{\circ}{A}^{\alpha})_{:\alpha} + b_{\alpha\beta}(\underset{\circ}{V}^{\alpha\beta} + \underset{\circ}{A}^{\alpha\beta}) + \underset{\circ}{N}^{3} = 0 \\
&(\underset{\circ}{W}^{\beta\alpha} + \underset{\circ}{B}^{\beta\alpha})_{:\beta} - \underset{\circ}{Q}^{\alpha} - b^{\alpha}_{\beta}\underset{\circ}{A}^{\beta} + \underset{\circ}{E}^{\alpha} + \underset{\circ}{M}^{\alpha} = 0 \\
&(\underset{\circ}{W}^{\alpha 3} + \underset{\circ}{B}^{\alpha})_{:\alpha} + b_{\alpha\beta}(\underset{\circ}{W}^{\alpha\beta} + \underset{\circ}{B}^{\alpha\beta}) - \underset{\circ}{N}^{33} + \underset{\circ}{E}^{3} + \underset{\circ}{M}^{3} = 0
\end{aligned}\right\} \text{ on } AXT \tag{83}$$

with the denotations by

$$\underset{\circ}{V}^{\alpha\beta} = \underset{\circ}{N}^{\alpha\beta} - b^{\beta}_{\sigma}\underset{\circ}{M}^{\alpha\sigma}, \quad \underset{\circ}{V}^{\alpha 3} = \underset{\circ}{Q}^{\alpha}, \quad \underset{\circ}{W}^{\alpha\beta} = \underset{\circ}{M}^{\alpha\beta} - b^{\beta}_{\sigma}\underset{\circ}{K}^{\alpha\sigma}, \quad \underset{\circ}{W}^{\alpha 3} = \underset{\circ}{R}^{\alpha} \tag{84}$$

and

$$\begin{aligned}
&\underset{\circ}{A}^{\alpha\beta} = \underset{\circ}{N}^{\alpha\sigma}(\underset{\circ}{v}^{\beta}{}_{:\sigma} - b^{\beta}_{\sigma}\underset{\circ}{v}^{3}) + \underset{\circ}{M}^{\alpha\sigma}(\underset{\circ}{w}^{\beta}{}_{:\sigma} - b^{\beta}_{\sigma} w^{\circ}_{3}) + \underset{\circ}{Q}^{\alpha}\underset{\circ}{w}^{\beta} \\
&\underset{\circ}{A}^{\alpha} = \underset{\circ}{N}^{\alpha\beta}(v^{\circ}_{3,\beta} + b^{\sigma}_{\beta} v^{\circ}_{\sigma}) + \underset{\circ}{M}^{\alpha\beta}(w^{\circ}_{3,\beta} + b^{\sigma}_{\beta} w^{\circ}_{\sigma}) + \underset{\circ}{Q}^{\alpha}\underset{\circ}{w}^{3} \\
&\underset{\circ}{B}^{\alpha\beta} = \underset{\circ}{M}^{\alpha\sigma}(\underset{\circ}{v}^{\beta}{}_{:\sigma} - b^{\beta}_{\sigma}\underset{\circ}{v}^{3}) + \underset{\circ}{K}^{\alpha\sigma}(\underset{\circ}{w}^{\beta}{}_{:\sigma} - b^{\beta}_{\sigma} w^{\circ}_{3}) + \underset{\circ}{R}^{\beta}\underset{\circ}{w}^{\alpha} \\
&\underset{\circ}{B}^{\alpha} = \underset{\circ}{M}^{\alpha\beta}(v^{\circ}_{3,\beta} + b^{\sigma}_{\beta} v^{\circ}_{\sigma}) + \underset{\circ}{K}^{\alpha\beta}(w^{\circ}_{3,\beta} + b^{\sigma}_{\beta} w^{\circ}_{\sigma}) + \underset{\circ}{R}^{\alpha}\underset{\circ}{w}^{3} \\
&\underset{\circ}{E}^{\alpha} = -(\underset{\circ}{R}^{\beta}\underset{\circ}{w}^{\circ}{}_{:\beta} + \underset{\circ}{N}^{33}\underset{\circ}{w}^{\alpha}) - \underset{\circ}{Q}^{\beta}(\underset{\circ}{v}^{\alpha}{}_{:\beta} - b^{\alpha}_{\beta}\underset{\circ}{v}^{3} - b^{\alpha}_{\beta}\underset{\circ}{w}^{3}) \\
&\underset{\circ}{E}^{3} = -(\underset{\circ}{R}^{\alpha}\underset{\circ}{w}^{3}{}_{:\alpha} + \underset{\circ}{N}^{33}\underset{\circ}{w}^{3}) - \underset{\circ}{Q}^{\alpha}(v^{\circ}_{3,\alpha} + b^{\beta}_{\alpha} v^{\circ}_{\beta})
\end{aligned} \tag{85}$$

in the notation of (43)-(46). Also, the distributions of strain are obtained as

$$S^{o}_{ij}(x^k)=e^{o}_{ij}(x^\alpha)+x^3\varepsilon^{o}_{ij}(x^\alpha)+(x^3)^2\gamma^{o}_{ij}(x^\alpha) \tag{86}$$

with

$$\begin{aligned}
e^{o}_{\alpha\beta}&=\frac{1}{2}\Big[v^{o}_{\alpha:\beta}+v^{o}_{\beta:\alpha}-2b_{\alpha\beta}v^{o}_{3}+v_{o}{}^{\delta}{}_{:\alpha}v^{o}{}_{\delta:\beta}-b^{\delta}_{\beta}v^{o}_{3}v^{o}{}_{\delta:\alpha}-b^{\delta}_{\alpha}v^{o}_{3}v^{o}{}_{\delta:\beta}\\
&\quad+c_{\alpha\beta}(v^{o}_{3})^2+v^{o}_{3,\alpha}v^{o}_{3,\beta}+b^{\delta}_{\beta}v^{o}_{\delta}v^{o}_{3,\alpha}+b^{\delta}_{\alpha}v^{o}_{\delta}v^{o}_{3,\beta}+b^{\delta}_{\beta}b^{\sigma}_{\alpha}v^{o}_{\delta}v^{o}_{\sigma}\Big]\\
\varepsilon^{o}_{\alpha\beta}&=\frac{1}{2}\Big[w^{o}_{\alpha:\beta}+w^{o}_{\beta:\alpha}-2b_{\alpha\beta}w^{o}_{3}-b^{\delta}_{\alpha}v^{o}{}_{\delta:\beta}-b^{\delta}_{\beta}v^{o}{}_{\delta:\alpha}+2c_{\alpha\beta}v^{o}_{3}+v_{o}{}^{\delta}{}_{:\alpha}w^{o}{}_{\delta:\beta}\\
&\quad+v_{o}{}^{\delta}{}_{:\beta}w^{o}{}_{\delta:\alpha}-b^{\delta}_{\beta}w^{o}_{3}v^{o}{}_{\delta:\alpha}-b^{\delta}_{\alpha}w^{o}_{3}v^{o}{}_{\delta:\beta}-b^{\delta}_{\beta}v^{o}_{3}w^{o}{}_{\delta:\alpha}-b^{\delta}_{\alpha}v^{o}_{3}w^{o}{}_{\delta:\beta}\\
&\quad+2c_{\alpha\beta}v^{o}_{3}w^{o}_{3}+v^{o}_{3,\alpha}w^{o}_{3,\beta}+v^{o}_{3,\beta}w^{o}_{3,\alpha}+b^{\delta}_{\beta}w^{o}_{\delta}v^{o}_{3,\alpha}+b^{\delta}_{\alpha}w^{o}_{\delta}v^{o}_{3,\beta}+b^{\sigma}_{\beta}v^{o}_{\sigma}w^{o}_{3,\alpha}\\
&\quad+b^{\sigma}_{\alpha}v^{o}_{\sigma}w^{o}_{3,\beta}+b^{\delta}_{\beta}b^{\sigma}_{\alpha}v^{o}_{\delta}w^{o}_{\sigma}+b^{\delta}_{\alpha}b^{\sigma}_{\beta}v^{o}_{\delta}w^{o}_{\sigma}\Big] \qquad (87)\\
\gamma^{o}_{\alpha\beta}&=\frac{1}{2}\Big[-b^{\delta}_{\alpha}w^{o}{}_{\delta:\beta}-b^{\delta}_{\beta}w^{o}{}_{\delta:\alpha}+2c_{\alpha\beta}w^{o}_{3}+w^{o}{}_{\delta:\beta}w^{o\delta}{}_{:\alpha}-b^{\delta}_{\beta}w^{o}{}_{\delta:\alpha}w^{o}_{3}\\
&\quad-b^{\delta}_{\alpha}w^{o}{}_{\delta:\beta}w^{o}_{3}+c_{\alpha\beta}(w^{o}_{3})^2+w^{o}_{3,\alpha}w^{o}_{3,\beta}+b^{\delta}_{\beta}w^{o}_{\delta}w^{o}_{3,\alpha}+b^{\delta}_{\alpha}w^{o}_{\delta}w^{o}_{3,\beta}+b^{\delta}_{\beta}b^{\sigma}_{\alpha}w^{o}_{\sigma}w^{o}_{\delta}\Big]\\
e^{o}_{\alpha3}&=\frac{1}{2}(w^{o}_{\alpha}+v^{o}_{3,\alpha}+b^{\beta}_{\alpha}v^{o}_{\beta}+w^{o}_{\beta}v^{o\beta}_{3}{}_{:\alpha}-b^{\beta}_{\alpha}w^{o}_{\beta}v^{o}_{3}+w^{o}_{3}v^{o}_{3,\alpha}+b^{\beta}_{\alpha}v^{o}_{\beta}w^{o}_{3})\\
\varepsilon^{o}_{\alpha3}&=\frac{1}{2}(w^{o}_{3,\alpha}+w^{o}_{\beta}w^{\beta}_{o}{}_{:\alpha}+w^{o}_{3}w^{o}_{3,\alpha}),\quad e^{o}_{33}=w^{o}_{3}-\frac{1}{2}\big[w^{o}_{\alpha}w^{\alpha}_{o}+(w^{o}_{3})^2\big]\\
&\varepsilon^{o}_{33}=\gamma^{o}_{i3}=0
\end{aligned}$$

the macroscopic constitutive relations as

$$\begin{aligned}
(N^{\alpha\beta}_{o},\ M^{\alpha\beta}_{o},\ K^{\alpha\beta}_{o})&=(\underset{\sim}{C}_{o},\ \underset{\sim}{C}_{1},\ \underset{\sim}{C}_{2})^{\alpha\beta kl}(\underset{\sim}{S}^{o}_{kl})^{T}\\
(Q^{\alpha}_{o},\ R^{\alpha}_{o})&=(\underset{\sim}{C}_{o},\ \underset{\sim}{C}_{1})^{\alpha3kl}(\underset{\sim}{S}^{o}_{kl})^{T},\quad N^{33}_{o}=(\underset{\sim}{C}_{o})^{33kl}(\underset{\sim}{S}^{o}_{kl})
\end{aligned} \tag{88a}$$

with

$$(\underset{\sim}{S}^{o}_{kl}) = (e^{o}_{kl},\ \varepsilon^{o}_{kl},\ \gamma^{o}_{kl}) \tag{88b}$$

the natural boundary conditions of initial tractions as

$$N^{\beta}_{*o}-\nu_{\alpha}(V^{\alpha\beta}_{o}+A^{\alpha\beta}_{o})=0,\quad N^{3}_{*o}-\nu_{\alpha}(V^{\alpha3}_{o}+A^{\alpha}_{o})=0 \tag{89}$$

$$M^{\beta}_{*o} - \nu_{\alpha}(W^{\alpha\beta}_{o} + B^{\alpha\beta}_{o}) = 0, \quad M^{3}_{*\ o} - \nu_{\alpha}(W^{\alpha 3}_{o} + B^{\alpha}_{o}) = 0 \quad \text{along } C_t X T_o \tag{89}$$

and

$$S^{j}_{*o} - q^{j}_{o} = 0 \text{ on } S_{uf} X T_o, \quad P^{j}_{*} - P^{j}_{o} = 0 \text{ on } S_{lf} X T_o \tag{90}$$

and those of mechanical displacements as

$$v^{o}_{i} - v^{o*}_{i} = 0, \quad w^{o}_{i} - w^{o*}_{i} = 0 \text{ along } C_u X T_o \tag{91}$$

In deriving (83)-(91), the resultants of initial state (43)-(46), the relations (30)-(33) and the identities (66) and (67) are considered.

In the foregoing analysis, an electroelastic shell theory for piezoceramics under a mechanical bias is systematically and consistently established via a variational procedure. The two-dimensional theory is constituted by the fields of mechanical displacements and electric potential (34), the distributions of infinitesimal strain and electric field (35) and (52)-(55), the macroscopic constitutive equations (57) and (58), the macroscopic stress equations of incremental motion (68) and charge equations of electrostatics (73), the natural boundary conditions (75)-(77) and (79), and the initial conditions (80) and (81) at the perturbed state; and also, by the fields of initial mechanical displacements (82) the distributions of initial finite strain (86) and (87), the macroscopic constitutive relations (88), the macroscopic stress equations of equilibrium (83) and the natural boundary conditions (89)-(91) at the unperturbed state of piezoceramic shell of uniform thickness. The complete set of two-dimensional electroelastic equations governs the finite static deformation of piezoceramic shell at the initial state, and then accommodates all the incremental types of extensional, thickness and flexural motions as well as their coupled motions of biased piezoceramic shell at the spatial state.

8 - SOME SPECIAL CASES

In the preceding sections, a shell theory is established for vibrations of piezoceramics under initial stresses in invariant form in the x^i-system of general convected coordinates. Thus, the shell theory is quite general and readily reducible to various special cases of engineering interest. Of special cases, those concerning with the material properties, kinematics and geometry of piezoceramic shell and those of biased piezoceramic composite shell and unbiased piezoceramic shell are pointed out. Also, a complete linearization of the resulting equations is given. The uniqueness in solutions of the linearized equations is studied by means of the positive definiteness of energies.

Thickness polarization - The constitutive relations (60) and (61) of biased piezoceramic shell that hold for all the linear piezoelectric materials are now specialized for the case of thickness polarization. In such a case, the direction of polarization coincides with the thickness axis-x^3, and the elastic and piezoelectric strain constants and the dielectric permittivity are axpressed by 12 independent constants, in lieu of 45 in the general case, as follows

$$[c^{pq}] = \begin{bmatrix} c^{11} & c^{13} & c^{12} & 0 & 0 & 0 \\ c^{13} & c^{33} & c^{13} & 0 & 0 & 0 \\ c^{12} & c^{13} & c^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c^{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c^{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c^{44} \end{bmatrix}, \quad c^{55} = \frac{1}{2}(c^{11} - c^{22}) \qquad (92a)$$

$$[e^{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & e^{16} \\ e^{31} & e^{33} & e^{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{16} & 0 & 0 \end{bmatrix}, \quad [\varepsilon^{ij}] = \begin{bmatrix} \varepsilon^{11} & 0 & 0 \\ 0 & \varepsilon^{33} & 0 \\ 0 & 0 & \varepsilon^{22} \end{bmatrix} \qquad (92b)$$

with

$$c^{pq} \equiv c^{ijkl}, \quad e^{ip} \equiv e^{ikl}$$

(ij) or $(kl) = 11,\ 22,\ 33,\ 23$ or $32,\ 31$ or $13,\ 12$ or 21 (92c)

(p) or $(q) = 1,\ 2,\ 3,\quad 4,\quad 5,\quad 6$

which should be replaced by those in (60) and (61).

Similarly, the case of polarization in other directions can be taken up in the constitutive relations of piezoceramic shell.

Kinematics - By invoking the Kirchhoff-Love hypothesis of elastic shells [81], namely,

$$w_\alpha=-(v_{3,\alpha}+b^\sigma_\alpha v_\sigma), \quad w_3=0,$$
$$\overset{o}{w}_\alpha=-(\overset{o}{v}_{3,\beta}+b^\sigma_\beta \overset{o}{v}_\sigma)(\delta^\beta_\alpha+v^\beta_{o:\alpha}), \quad \overset{o}{w}_3=-\frac{1}{2}\overset{o}{v}_{3,\alpha}v^\alpha_{o3}, \tag{93}$$

in (34) and (82), one obtains, approximately, the classical case,

$$\gamma_{i3}=\overset{o}{\gamma}_{i3}\simeq 0 \tag{94}$$

Besides, in consistent with (94), a restriction of the form

$$\psi=0$$

should be included in (34). The inclusion of (93)-(95) leads to vanishing transverse shear and normal strains in the resulting equations of biased piezoceramic shell.

Geometry - The formulation being in invariant form the resulting equations can be readily expressed in any particular system of coordinates most suitable for the geometrical configuration of piezoceramic shell under consideration (e.g., cylindrical and spherical shells or shallow shells). Besides, in the absence of curvature effect,

$$b^\alpha_\beta=0; \quad \mu^\alpha_\beta=\delta^\alpha_\beta, \quad \mu=1; \quad \mu_n=I_n, \quad K_m=K_g=0 \tag{96}$$

and hence, the governing equations of piezoceramic shell are reduced to those of piezoceramic plate of arbitrary shape. With these simplifications, the two-dimensional equations of incremental motion in the form

$$N^{\alpha\beta}{}_{:\alpha}+(N^{\alpha\sigma}_o v^\beta_{:\sigma}+M^{\alpha\sigma}_o w^\beta_{:\sigma}+Q^\alpha_o w^\beta)_{:\alpha}+1^\beta+1^\beta_o-\rho A^\beta=0$$
$$Q^\alpha{}_{:\alpha}+(Q^\alpha_o w_3+N^{\alpha\beta}_o v_{3,\beta}+M^{\alpha\beta}_o w_{3,\beta})_{:\alpha}+1^3+1^3_o-\rho A^3=0$$
$$M^{\alpha\beta}{}_{:\alpha}+(M^{\alpha\sigma}_o v^\beta{}_{:\sigma}+K^{\alpha\sigma}_o w^\beta{}_{:\sigma}+R^\alpha_o w^\beta)_{:\alpha}-Q^\beta-(Q^\alpha_o v^\beta{}_{:\alpha}+N^{33}_o w^\beta+R^\alpha_o w^\beta{}_{:\alpha})$$
$$+m^\beta+m^\beta_o-\rho B^\beta=0$$
$$R^\alpha{}_{:\alpha}+(M^{\alpha\beta}_o v_{3,\beta}+K^{\alpha\beta}_o w_{3,\beta}+R^\alpha_o w_3)_{:\alpha}-N^{33}-(Q^\alpha_o v_{3,\alpha}+R^\alpha_o w_{3,\alpha}+N^{33}_o w_3)$$
$$+m^3+m^3_o-\rho B^3=0 \qquad \text{on } AXT \tag{97}$$

with

$$(q^j, p^j) = t^{3j} \quad \text{at } (x^3 = h, -h)$$
$$(r_o^\alpha, s_o^\alpha) = t^{3\beta}(v^\alpha{}_{:\beta} - x^3 w^\alpha{}_{:\beta}) + t_o^{33} w^\alpha \quad \text{at} (x^3 = h, -h) \tag{98}$$
$$(r_o^3, s_o^3) = t_o^{3\alpha}(v_{3,\alpha} + x^3 w_{3,\alpha}) + t_o^{33} w_3 \quad \text{at} (x^3 = h, -h)$$

the natural boundary conditions of traction as

$$\left.\begin{aligned} &N_*^\alpha - \nu_\beta (N^{\beta\alpha} + N_o^{\beta\sigma} v^\alpha{}_{:\sigma} + M_o^{\beta\sigma} w^\alpha{}_{:\sigma} + Q_o^\beta w^\alpha) = 0 \\ &N_*^3 - \nu_\beta (Q^\beta + Q_o^\beta w_3 + N_o^{\beta\alpha} v_{3,\alpha} + M_o^{\beta\alpha} w_{3,\alpha}) = 0 \\ &M_*^\alpha - \nu_\beta (M^{\beta\alpha} + M_o^{\beta\sigma} v^\alpha{}_{:\sigma} + K_o^{\beta\sigma} w^\alpha{}_{:\sigma} + R_o^\beta w^\alpha) = 0 \\ &M_*^3 - \nu_\alpha (R^\alpha + R_o^\alpha w_3 + M_o^{\alpha\beta} v_{3,\beta} + K_o^{\alpha\beta} w_{3,\beta}) = 0 \end{aligned}\right\} \quad \text{on } C_t XT \tag{99}$$

and

$$S_*^j - (q^j + r_o^j) = 0 \text{ on } S_{uf} XT, \quad P_*^j - (p^j + s_o^j) = 0 \text{ on} s_{1f} XT \tag{100}$$

with (98), the distribution of strain as

$$\begin{aligned} &e_{\alpha\beta} = \frac{1}{2}(v_{\alpha:\beta} + v_{\beta:\alpha}), \quad e_{\alpha 3} = \frac{1}{2}(w_\alpha + v_{3,\alpha}), \quad e_{33} = w_3 \\ &\varepsilon_{\alpha\beta} = \frac{1}{2}(w_{\alpha:\beta} + w_{\beta:\alpha}), \quad \varepsilon_{\alpha 3} = \frac{1}{2} w_{3,\alpha}, \quad \varepsilon_{33} = 0; \quad \gamma_{ij} = 0 \end{aligned} \tag{101}$$

and (34)-(37), (47)-(49), (55), (57)-(63), (73), (75) and (79)-(81) are recorded at the perturbed state, and (43)-(46) and (82)-(91) at the unperturbed state for a piezoceramic plate under initial stresses.

Piezoceramic composite shell - The two-dimensional equations of biased piezoceramic shell can also accomodate the incremental motions of a piezoceramic composite shell with N layers for the case when the mechanical displacements and electric potential fields vary linearly in the form

$$\begin{aligned} &\bar{u}_\alpha^{(m)}(x^i,t) = v_\alpha^{(m)}(x^\beta,t) - x^3 w^{(m)}{}_{,\alpha}, \quad \bar{u}_3^{(m)}(x^i,t) = w^{(m)}(x^\alpha,t) \\ &\phi^{(m)}(x^i,t) = \varphi^{(m)}(x^\alpha,t); \quad m = 1, 2, \ldots, N \end{aligned} \tag{102}$$

within the concept of the effective modulus of composites (e.g., [98]). The piezoceramic shell consists of two perfectly conducting electrodes at its faces and (N-2) layers between them. Each constituent of piezoceramic shell may possesss distinct but uniform thickness, curvature and electromechanical properties. Also, the constituents are

well attached one another, and hence the relative deformations are prevented at their interfaces. Thus, (102) yields the fields as

$$v_\alpha^{(m)} = v_\alpha, \quad w^{(m)} = w; \quad \Phi^{(m)} = \psi \tag{103}$$

due to the continuity of mechanical displacements and electric potential at the interfaces of adjacent layers. In view of (103), the resulting equations account for the incremental motions of piezoelectric laminae provided that (2h) is considered as its overall thickness, and accordingly the integrations are carried out, and also, the simplifications implied by (103) are taken into account. A detailed analysis of biased piezoceramic laminae is beyond the scope of this paper and will be taken up in a forthcoming study [99].

Unbiased piezoceramic shell - When the terms involving the incremental motions are omitted, that is, the terms indicated by a zero index together with the electrical terms are retained only, a dynamic theory is obtained for the finite motions of piezoceramic shell without a bias. The two-dimensional theory is both geometrically and physically nonlinear in view of the distribution of strain (86) together with (25) and the constitutive relations (57) and (58). The governing equations of the fully nonlinear theory are (34b), (43)-(49), (55), (57) where the resultants should be replaced by those with a zero index, (58), (59), (73), (79)-(82), (83) with the acceleration terms as in (68), (84)-(87) and (89)-(91). In virtue of the field of mechanical displacements (82), this shear deformable theory accounts for the motions of piezoceramic shell subjected to large displacement gradients and large angles of rotation. The finite theory contains some of earlier theories which were always geometrically linear, as special cases. Another important special case is found by taking a partially nonlinear version of the strain-mechanical displacement relations (25) in the form

$$\begin{gathered} S^o_{\alpha\beta} = \frac{1}{2}(u^o_{\alpha;\beta} + u^o_{\beta;\alpha} + u^o_{3,\alpha} u^o_{3,\beta}), \quad S^o_{\alpha 3} = \frac{1}{2}(u^o_{\alpha;3} + u^o_{3;\alpha}), \\ S^o_{33} = u^o_{3;3} \end{gathered} \tag{104}$$

in conjunction with (82). The use of (104) in derivation yields the governing electroelastic equations appropriate to a refined theory of piezoceramic shell of a von-Kārmān type [81]. Moreover, a complete linearization by discarding the terms of initial state and using the linear constitutive relations (61) and (88) leads to a fully linear theory of piezoceramic shell. In this case, the stress equations of motion are given by

$$
\left.\begin{aligned}
&(N^{\alpha\beta}-b^{\beta}_{\sigma}M^{\alpha\sigma})_{:\alpha}+b^{\beta}_{\alpha}Q^{\beta}+l^{\beta}-\rho A^{\beta}=0\\
&(M^{\alpha\beta}-b^{\beta}_{\sigma}K^{\alpha\sigma})_{:\alpha}-Q^{\beta}+m^{\beta}-\rho B^{\beta}=0\\
&Q^{\alpha}_{:\alpha}+b_{\alpha\beta}(N^{\alpha\beta}-b^{\beta}_{\sigma}M^{\alpha\sigma})+l^{3}-\rho A^{3}=0\\
&R^{\alpha}_{:\alpha}-N^{33}+b_{\alpha\beta}(M^{\alpha\beta}-b^{\beta}_{\sigma}K^{\alpha\sigma})+m^{3}-\rho B^{3}=0
\end{aligned}\right\}\quad \text{on } AXT \tag{105}
$$

and the associated boundary conditions by

$$
\left.\begin{aligned}
&N^{\alpha}_{*}-\nu_{\beta}(N^{\alpha\beta}-b^{\alpha}_{\sigma}M^{\beta\sigma})=0,\quad N^{3}_{*}-\nu_{\alpha}Q^{\alpha}=0\\
&M^{\alpha}_{*}-\nu_{\beta}(M^{\alpha\beta}-b^{\alpha}_{\sigma}K^{\beta\sigma})=0,\quad M^{3}_{*}-\nu_{\alpha}R^{\alpha}=0
\end{aligned}\right\}\quad \text{along } C_{t}XT \tag{106}
$$

and

$$
S^{j}_{*}-q^{j}=0 \text{ on } S_{uf}XT,\quad P^{j}_{*}-p^{j}=0 \text{ on } S_{lf}XT \tag{107}
$$

In addition to (105)-(107), the charge equations of electrostatics (73), the electrical boundary conditions (79), the linear constitutive relations (60) and (61), the fields of mechanical displacements and electric potential (34), the distributions of strain and electric field (52) and (55) and the initial conditions (80) and (81) constitute the governing equations of the linear theory. The results of the aforementioned special cases, with various applications will be reported in detail in a separate memoir.

The governing equations of the linear theory of piezoceramic shell have a unique solution under the mixed-boundary and initial conditions (79)-(81), (106) and (107). To establish the uniqueness of solutions, the existence of two possible sets of solutions identified by prime and double prime, and their differences by $u_i(=u_i'-u_i'')$ and alike is considered. In terms of the difference variables, the internal energy of piezoceramic shell is expressed by

$$
\Omega=\int_{B}\frac{1}{2}(t^{ij}s_{ij}+D^{i}E_{i})dV \tag{108}
$$

By taking time differentiation, considering (22) and inserting (3) and (4) into this equation, the rate of the internal energy is given by

$$
\dot{\Omega}=\int_{A}dA\int_{H}(t^{ij}\dot{u}_{i;j}-D^{i}\dot{\phi}_{,i})\mu dx^{3} \tag{109}
$$

Substituting (34) into (109), then carrying out the integration with respect to x^3 and applying (31), the rate of the internal energy is obtained as

$$\dot{\Omega} = \int_A \{[V^{\alpha\beta}\dot{v}_{\alpha:\beta} + W^{\alpha\beta}\dot{w}_{\alpha:\beta} - b_{\alpha\beta}(V^{\alpha\beta}\dot{v}_3 + W^{\alpha\beta}\dot{w}_3) + V^{\alpha 3}\dot{v}_{3,\alpha} + W^{\alpha 3}\dot{w}_{3,\alpha}$$
$$b^{\alpha}_{\beta}Q^{\beta}\dot{v}_{\alpha} + Q^{\alpha}\dot{w}_{\alpha} + N^{33}\dot{w}_3] - (F^{\alpha}\dot{\phi}_{,\alpha} + G^{\alpha}\dot{\psi}_{,\alpha} + F^3\dot{\psi})\}dA \qquad (110)$$

in terms of the *denotations* (36), (47) *and* (69). Analogously, the rate of the kinetic energy of piezoceramic shell defined by

$$\Sigma = \int_A dA \int_H \frac{1}{2}\rho\dot{u}^i\dot{u}_i\mu dx^3 \qquad (111)$$

is found in terms of the acceleration resultants (37) as

$$\Sigma = \int_A \rho(A^i\dot{v}_i + B^i\dot{w}_i)dA \qquad (112)$$

where (30) and (34) are used. Besides, in view of (73) and (105), the equation as follows

$$\int_T dt \int_A [(V^i\dot{v}_i + W^i\dot{w}_i) - (F^{\alpha}{}_{:\alpha} + c - d)\dot{\phi} - (G^{\alpha}{}_{:\alpha} - F^3 + e)\dot{\psi}]dA = 0 \qquad (113)$$

is formed. This can be transformed, by applying the divergence theorem, considering (110) and (112), integrating over time and assembling all terms, to

$$\int_A (\Sigma+\Omega)\Big|_{t_o}^{t_1} dA = \int_T dt\left(\oint_C \nu_\alpha \Gamma^\alpha ds + \int_A \Gamma dA\right) \qquad (114a)$$

with the definitions by

$$\Gamma = (V^{\alpha i}\dot{v}_i + W^{\alpha i}\dot{w}_i) - (F^{\alpha}\dot{\phi} + G^{\alpha}\dot{\psi})$$
$$\Gamma = (l^i\dot{v}_i + m^i\dot{w}_i) - [(c-d)\dot{\phi} + e\dot{\psi}] \qquad (114b)$$

The kinetic and potential energy densities are positive-definite by definition, and initially zero; so that the kinetic and potential energies of piezoceramic shell and also Σ and Ω, calculated in terms of the difference variables possess the same properties. Thus, if Γ^α and Γ are zero in (114), one reads

$$\Sigma(t_1) = \Omega(t_1) = \Sigma(t_o) = \Omega(t_o) = 0 \qquad (115)$$

which implies a trivial solution for the difference set of solutions, that is, the two solutions are equal. The boundary and initial conditions (79)-(81), (106) and (107) as well as to specify one member of each product in (114b) make the right-hand side of (114a) zero, and hence, they are evidently sufficient to ensure the uniqueness of solutions.

9 - CONCLUSIONS

The main result of this paper is a shell theory in invariant form for small motions superposed upon a static, finite deformation of piezoceramics subjected to mechanical biasing stresses. The set of two-dimensional, approximate, electro-elastic governing equations of the shell theory is established by means of the unified variational principles (20) and (24) together with the fields of mechanical displacements and electric potential (34) and (82). The governing equations in a complete Lagrangian description are given for the unperturbed static state and the perturbed dynamic state of piezoceramic shell coated with very thin, perfectly conducting electrodes. The shell theory accounts for all the incremental types of extensional, thickness and flexural as well as their coupled motions of, and also, for the initial, finite, static deformation of, piezoceramic shell of uniform thickness. The fields (34) and (82) which are chosen as a basis of systematic and consistent derivation of the shell theory take into account all the significant mechanical and electrical effects, and they are able to predict the influence of biasing stresses on the dynamic response of piezoceramic shell. The two-dimensional, variational versions of the governing equations (51), (54), (56), (65), (72), (74) and (78) provide an appropriate basis for numerical direct solutions, for instance, based on the Rayleigh-Ritz procedure or the finite element method [100]. The unified variational principles are deduced from the principle of virtual work by augmenting it through the dislocation potentials and Lagrange undetermined multipliers. As their Euler-Lagrange equations, the variational principles yield all the fundamental equations of the initial state and those of the spatial state but its initial conditions, of piezoceramic strained shell. The variational principles do agree with those extracted from Hamilton's principle in Cartesian coondinates [36] and contain certain known results (e.g., [28], [32]-[35], [82] [88], and those cited in [101]), as special cases.

The shell theory incorporating the geometrical and physical nonlinearities is quite general, and hence, it leads to a variety of intermediate theories by considering special motions, geometry and material of piezoceramic shell. The resulting equations of shell theory may be expressed in any particular system of coordinates most suitable for the geometrical configuration of piezoceramic shells and plates at hand. A fully nonlinear theory of piezoceramic shell is stated in the previous section; this includes the results

reported in [60], [63], [65] and [82] for the case when the effect of geometrical nonlinearity is omitted and also the curvature effect is absent. The electroelastic governing equations of piezoceramic shell are shown to be applicable for the incremental motions of a piezoceramic laminae, and they are explicitly stated for a biased piezoceramic plate of arbitrary shape and an unbiased piezoceramic shell. In particular, the fully linear theory of piezoceramic shell which does agree with the known results is described, and then, the uniqueness is examined in solutions of its governing equations. The sufficient boundary and initial conditions are enumerated for the uniqueness by use of the classical energy arguments (cf., [102] for elastic shells and [89] for thermopiezoelectric plates). Similar results for the uniqueness may be obtained by means of the logarithmic convexity arguments ([103], and, for instance, [14] for elastic shells).

Final remarks are in order concerning extensions and applications of the shell theory presented. The shell theory may provide an appropriate basis for approaching to the stability of piezoceramic shell. Another theory may be established for piezoceramic viscoelastic shells by replacing the elastic stiffenesses of piezoceramic shell by their corresponding convolution integrals. Also, the shell theory may be extended so as to incorporate the mechanical effect of electrical coatings as in [60],[82], the effect of couple stresses (e.g., [104]) and the thermal effect (e.g., [105]), and especially, it may be developed for piezoceramics under a biasing electric and thermal field, as investigated in [29] and [106] for elastic plates, and even under a magnetic field [107]. In closing, some of the above mentioned points of importance is addressed in [101].

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CHAPTER 6

DYNAMICS OF PIEZOELECTRIC LAMINAE UNDER A BIAS

This paper is addressed to the macromechanical analysis of dynamics of a piezoelectric laminae under a mechanical bias within the effective stiffness concept of laminated composites. The piezoelectric laminae consists of arbitrary numbers of perfectly bonded layers, each with a distinct but uniform thickness, curvature and eletromechanical properties, and it is coated with very thin electrodes on both its faces. First, the fundamental equations of piezoelectric strained medium are expressed by the Euler-Lagrange equations of a unified variational principle. Secondly, a set of two-dimensional, approximate equations of the piezoelectric laminae is consistently established. Thirdly, a direct method of solution is indicated for the macromechanical analysis and certain special cases are considered. The governing equations are derived in invariant Lagrangian form and accommodate all the types of motions of the biased piezoelectric laminae. All the significant effects, both mechanical and electrical, are taken into account.

1- INTRODUCTION

Laminae or multilayer type of structural elements was appreciated only relatively recently due to their significant improvements in piezoelectric properties for ultrasonic technology. The features and applications of piezoelectric layered and/or composite elements and the basic ideas underlying their sum and product properties are available[1-4]. To predict dynamic response of this type of structural elements, there basically exist two types of macromechanical models: the effective modulus model and the effective stiffeness model. The former model [5,6] replaces a laminae by a representative homogeneous medium with the aid of averaged material constants of laminae constituents. This model, although it is relatively simple, omits the mutual coupling of layers, and it is generally suitable for a rather broad class of static response of laminae. The effective stiffeness model combines both the physical and geometrical properties of laminae constituents and incorporates all their essential electromechanical features, and it accounts for dynamic response of laminae as well.

Within the frame of this model [7], this paper describes a macromechanical analysis of piezoelectric laminae under a state of mechanical bias.

Extension of classical models (for instance, Lagrange's or Kármán's model of plates and Love-Kirchhoff's model of (shells) to piezoelectric laminae leads always to an effective modulus model, and hence it disregards the electromechanical interactions between adjacent layers. On the basis of classical models, the macroscopic relations of electroelasticity weere derived for multilayer piezoelectric plates and shells; their steady-state vibrations were reported in some special cases [8-10] as well. Parton and Senik [8] derived the macroscopic equations of multilayer piezoceramic shells with thickness polarization of the layers. Likewise, Karnaukhov and Kirichok [10-12] constructed the governing equations of laminated piezoelectric plates and shells by taking into account the geometrical nonlinearity and, in particular, the effect of viscosity and temperature. Evseichik, Rudnitskii and Shul'ga [13] derived the electroelastic equations for the vibrations of a shell that is inhomogeneous in thickness and has piezoelectric layers. Moreover, the thermomechanical behavior of multilayered piezoceramic shells with thickness polarization was treated under harmonic excitation by Motovilovets and Gololobov [14]. Mention should also be made of a theory of vibrations of coated, thermopiezoelectric laminae in which the effects of elastic stiffness of, and the interactions between, layers of the laminae and its electrodes were all included [7]. On the other hand, Holland and EerNisse [15] described the design and analysis of laminae types of piezoelectric bars, disks and plates by means of Green's function technique. Auld and his coworkers [16,17] developed a Floquet theory of wave propagation in periodic composites that was shown to agree wel with experiment. Buğdaycı and Bogy [18,19] derived a theory for high frequency motions of piezoelectric layers, including some applications, as did Lee and Moon [20] for low frequency motions of piezoelectric laminated plates. Moreover, a general transfer matrix description of arbitrarily layered piezoelectric structures was obtained [21].

Biasing stress or strain and/or electric field is a new design feature and demand in piezoelectric devices for ultrasonic application in control engineering. The presence of a biasing state induced by external perturbations like thermal, mechanical and electrical fields and even magnetic fields can significantly affect the dynamic response of structural elements (e.g., rods [22], plates [23,24] and

shells [25,26] and the characteristics of BAW and SAW (e.g., [27-31] and references therein). In a biased solid medium, the linear theory of electroelasticity is in need of modification so as to govern its motions. This fact was widely recognized, and taken up by many investigators in electroelasticity. Tiersten [32] derived a properly invariant set of the nonlinear fundamental equations, including thermal effects by means of a systematic use of the axioms of continuum phsics. From these general equations, he and Baumhauer [33] established the differential electroelastic equations for small dynamic fields superposed on a static biasing state of solid medium, and also, for intrinsically nonlinear fields. Moreover, the fundamental equations of a biased piezoelectric medium were expressed as the Euler-Lagrange equations of some variational principles [34,35].

The aim of the present paper is (i) to present a variational formulation of the fundamental equations of piezoelectric medium under a mechanical bias, and using this together with a linear representation for the field variables, (ii) to derive the two-dimensional, approximate governing equations for all the types of incremental motions of piezoelectric laminae under a bias, and then (iii) to describe a direct method of solution for the incremental motions, to indicate some special cases and also to consider the fully linearized equations of piezoelectric laminae.

Specifically, a definition of the notation to be used herein is given in the rest of this section and the content of the paper is as follows. In the first part of the paper, a unified variational principle is formulated by extending the principle of virtual work through Friedrich's transformation in Section 2. In the second part of the paper, presented in Sections 3-6, by use of Mindlin's method of reduction, the set of two-dimensional, approximate equations is consistently derived for the incremental motions of piezoelectric laminae under a static, finite, mechanical bias. The geometry of piezoelectric laminae region is described in Section 3. In Section 4, a linear representation in the thickness coordinate of piezoelectric laminae is introduced for the fields of incremental mechanical displacements and electric potential which are chosen as a starting point of derivation. Also, in accordance with the linear representation, various resultant quantities averaged over the thickness of laminae are defined. The distributions of incremental strain and quasi-static electric field are given and the macroscopic constitutive equations of piezoelectric laminae are obtained in Section 5. The two-dimensional, approximate governing

equations and the associated boundary and initial conditions for the motions of piezoelectric laminae are deduced from the three-dimensional equations of piezoelectricity by use of the unified variational principle together with the series representation of field variables in Section 6. Alternatively, a direct method of solution is indicated in investigating the incremental motions of piezoelectric strained laminae in Section 7. Special motions, geometry and material are treated and the fully linearized governing equations, including the uniqueness of their solutions, are pointed out in Section 8. The last section is devoted to the concluding remarks.

N o t a t i o n - In the paper, standard tensor notation is freely used in a Euclidean 3-space E. Accordingly, Einstein's summation convention is implied over all repeated Latin (1,2,3) and Greek indices (1,2) that stand for space and surfaca tensors, respectively, unless they are put within parantheses. In the space E, a fixed, right-handed system of geodesic normal convected coordinates is identified by the x^i-system. All the field quantities are used in Lagrangian formulation, and a quantity in the initial state is designated by a zero index and a prescribed quantity by an asterisk. A superposed dot stands for time differentiation, a comma for partial differentiation with regard to the indicated space coordinate, and a semicolon and a colon for covariant differentiation with respect to the indicated coordinate, using the space and surface metrics, respectively. The index (m) which takes the values 1,2,....,N refers to the m-th constituent from the lower face of piezoelectric laminae, and, for instance, m=1 (or a prime') to the lower face electrode, m=2,3,...,N-1 to the layers and m=N (or a double prime") to the upper face electrode of laminae. Moreover, B(t) represents a regular, finite and bounded region B of the space E at time t, $\bar{B}(=B \cup \partial B)$ the closure of B, with its boundary surface ∂B, $\bar{B}XT$ the domain of definition for the functions of the space coordinates and time, $T=[t_o,t_1)$ the time interval, and $Z=[z-h,z+h]$ the thickness interval of a constituent.

2- PRINCIPLE OF VIRTUAL WORK FOR THE PIEZOELECTRIC MEDIUM UNDER A BIAS

To derive, in a systematic and consistent manner, lower order field equations and to directly provide their approximate solutions, variational principles were primarily developed by Mindlin, Tiersten and EerNisse for a piezoelectric medium, by Mindlin, Nowacki and the author for a thermopiezoelectric medium, and by the author for a piezoelectric medium, with small piezoelectric coupling and/or an internal surface of discontinuity and that under a mechanical bias. Hamilton's principle, the principle of virtual work and an experienced guess work were used in deducing the variational principles of piezoelectricity; a review of the subject was given in Refs [2, 35]. In order to render this paper to be self-contained, a unified variational principle is reformulated by extending the principle of virtual work through the dislocation potentials and Lagrange undetermined multipliers.

In the space E, referring to the x^i-system of general convected coordinates, a regular, finite and bounded region of piezoelectric elastic medium, B_o, with its boundary surface ∂B_o, under a state of mechanical static stresses is considered at its initial unperturbed or reference state at time $t=t_o$. This initial state which is taken to be self-equilibrating acquires its spatial (perturbed or final) state $B+\partial B$ by a small motion soperposed onto the finite, static deformation of piezoelectric region $B_o+\partial B_o$ at the time interval $T=[t_o,t_1)$. Now employing Lagrangian approach, the principle of virtual work is stated for the piezoelectric strained region at its spatial state as an assertion in the form.

$$-\int_B (T^{ij}\delta S_{ij}-D^i\delta E_i)dV + 1/2\delta\int_B \rho U^i U_i dV + \int_{\partial B}(T^i_{\star}\delta U_i+\sigma_{\star}\delta\phi)dS = 0 \qquad (1)$$

Here, $T^{ij}(=t_o^{ij}+t^{ij})$, t_o^{ij} and t^{ij} are the total, initial and incremental stress tensors; $S_{ij}=(s^o_{ij}+s_{ij})$, s^o_{ij} and s_{ij} the total, initial and incremental strain tensors; ρ the mass density of undeformed piezoelectric medium; $U_i(=u_i^o + u_i)$, u_i^o and u_i the total initial and incremental displacement vectors, $a^i(=\ddot{u}_i)$ the Lagrangian acceleration vector; T^i

$(=t_o^i + t^i)$, t_i^o and t^i the total, initial and incremental stress vectors; D^i the electric displacement vector, E^i the quasi-static electric field vector, $\sigma(=n_i D^i)$ the surface charge, Φ the electric potential and n_i the unit outward vector normal to a surface element. Substituting the gradient equations of the form.

$$S_{ij} = E_{ij} + 1/2\, U^k{}_{;i} U_{k;j} \quad \text{in } \bar{B}XT \tag{2}$$

$$E_{ij} = 1/2\,(U_{i;j} + U_{j;i}) \;, \qquad E_i = -\Phi_{,i}$$

into (2.1), applying the Green-Gauss transformation of integrals for the regular region $B+\partial B$, carrying out the indicated variations, and then integrating over T, one finally obtains a two-field variational principle for the piezoelectric biased medium as

$$\delta \mathcal{L}(u_i, \Phi) = \int_T dt \int_B (L^i \delta u_i + L\,\delta\Phi)\,dV$$
$$+ \int_T dt \int_{\partial B} (L_*^i \delta u_i + L_* \delta\Phi)\; dS = 0 \tag{3}$$

with the divergence equations of incremental motion by

$$L^j = (t^{ij} + t_o^{ik} u^j{}_{;k})_{;i} - \rho a^j = 0 \quad \text{in } \bar{B}XT \tag{4}$$

$$L = D^i{}_{;i} = 0 \quad \text{in } \bar{B}XT \tag{5}$$

and the associated natural boundary conditions by

$$L_*^i = \tau_*^j - n_i (t^{ij} + t_o^{ik} u^j{}_{;k}) = 0 \text{ on } \partial BXT \tag{6}$$

$$L_* = \sigma_* - n_i D^i = 0 \quad \text{on } \partial BXT \tag{7}$$

as its Euler- Lagrange equations. In deriving (3) the familiar relations between the stress tensors and the stress vectors, the stress equations of equilibrium and the associated boundary conditions, namely,

$$L_o^j = [t_o^{ik} (\delta_k^j + u_o^j{}_{;k})]_{;i} = 0 \quad \text{in} \quad \bar{B}XT$$
$$L_{o*}^j = t_{o*}^j - n_i t_o^{ik} (\delta_k^j + u_o^j{}_{;k}) = 0 \quad \text{on} \quad \partial BXT \tag{8}$$

are considered the usual argumentsare implied on the increments of field variables [36] and the axiom of conservation of mass is employed. Also, the constraint conditions of the form

$$\delta u_i = \delta\Phi = 0 \qquad \text{in } B(t_o) \quad \& \quad B(t_1) \tag{9}$$

are imposed.

To describe fully the motions of piezoelectric strained medium, the two-field variational principle (3) should be supplemented by the gradient equations (2), the constitutive equations in the form

$$t^{ij} = 1/2 \left(\frac{\partial \P}{\partial s_{ij}} + \frac{\partial \P}{\partial s_{ji}}\right), \quad D^i = -\frac{\partial \P}{\partial E_i} \tag{10}$$

where $\P(S_{ij}, E_i, t_o^{ij})$ stands for an electric enthalpy function which contains the initial stresses as parameters [37] in addition to (6), the boundary conditions as

$$u_i - u_i^* = 0 \text{ on } \partial B_u XT, \; \Phi - \Phi_* = 0 \text{ on } \partial B_\Phi XT \tag{11}$$

the initial conditions of the form

$$u_i(x^j, t_o) - v_i^*(x^j) = 0, \quad \dot{u}_i(x^j, t_o) - w_i^*(x^j) = 0$$

$$\Phi(x^i, t_o) - \Omega(x^i) = 0 \qquad \text{in } B(t_o) \tag{12}$$

and the constraint conditions (9). These constraint conditions prevent a free choice of trial functions, and hence, variational principles with as few constraints as possible become desirable in computation. Thus, all the constraint conditions but the initial conditions are relaxed through Friedrichs's transformation [34] and the initial conditions following Tiersten's [38] approach. The result is a unified variational principle by

$$\delta L\{\Lambda\} = \delta J^{\alpha i}_{\alpha i} + \delta I^{\alpha\beta}_{\alpha\beta} + \delta I^i_i = 0 \tag{13a}$$

with the admissible state

$$\Lambda = \{u_i, s_{ij}, t^{ij}, \tau^i; \Phi, E_i, D^i, \sigma\} \tag{13b}$$

and the denotations by

$$(\delta J^{11}_{11},\ \delta J^{12}_{12},\ \delta J^{13}_{13}) = \int_T dt \int_B (L^{i}\delta u_i,$$

$$L^{ij}\delta s_{ij},\ J_{ij}\delta t^{ij})\,dV$$

$$(\delta J^{21}_{21},\ \delta J^{22}_{22},\ \delta J^{23}_{23}) = \int_T dt \int_B (L\delta\phi,\ I^{i}\delta E_i,\ J_i\delta D^{i})\,dV \tag{14}$$

$$(\delta I^{11}_{11},\ \delta I^{12}_{12}) = \int_T dt\,(\int_{\partial B_t} L^{i}_{\star}\delta u_i\,dS;\ \int_{\partial B_u} J_{\star i}\delta\tau^{i}\,dS)$$

$$(\delta I^{21}_{21},\ \delta I^{22}_{22}) = \int_T dt\,(\int_{\partial B_\sigma} L_{\star}\delta\phi\,dS;\ \int_{\partial B_\phi} J_{\star}\delta\sigma\,dS)$$

and

$$[\delta I^{1}_{1},\ \delta I^{2}_{2}] = \int_B [\lambda^{i}\,\delta u_i(x^{j},t_o),\ \mu^{i}\delta\dot{u}_i(x^{j},t_o)]\,dV$$

$$\delta I^{3}_{3} = \int_B \lambda\delta\phi(x^{j},t_o)\,dV \tag{15}$$

Here,

$$L^{ij} = t^{ij} - 1/2\ (\frac{\partial\Psi}{\partial s_{ij}} + \frac{\partial\Psi}{\partial s_{ij}})$$

$$J_{ij} = s_{ij} - 1/2\ (u_{i;j} + u_{j;i}) \tag{16}$$

$$J_i = -\ (E_i + \phi_{,i}),\ I^{i} = -(D^{i} + \frac{\partial\Psi}{E_i})$$

and

$$J^{\star}_{i} = u_i - u^{\star}_{i},\quad J_{\star} = \sigma_{\star} - n_i D^{i} \tag{17}$$

and

$$\lambda^{i} = \rho[\dot{u}^{i}(x^{j},t_o) - w^{i}_{\star}(x^{j})]$$

$$\mu^{i} = \rho[u^{i}(x^{j},t_o) - v^{i}_{\star}(x^{j})] \tag{18}$$

$$\lambda = \phi(x^{j},t_o) - \phi_{\star}(x^{j})$$

and also those in (4)-(7) are defined. The variational principle should be modified for the linearized constitutive relations which implies the dislocation potentials by

$$L^{ij} = t^{ij} - (C^{ijkl}s_{kl} - C^{kij}\,E_k)$$
$$I^{i} = - D^{i} + (C^{ijk}s_{jk} + C^{ij}E_j) \qquad (19)$$

in lieu of those defined in (16). In (19), C^{ijkl}, C^{ijk} and C^{ij} are the elastic and piezoelectric strain constants and the dielectric permittivity of piezoelectric medium, with their usual symmetry properties [35].

Evidently, the unified eight-field variational principle (13) yields, as its Euler-Lagrange equations, all the fundamental differential equations of motion of piezoelectric strained media but the symmetry of stress tensor; and conversely, if the fundamental differential equations are satisfied, the unified variational principle is definitely verified.

The unified variational principle (13) recovers those [26,34] deduced from Hamilton's principle and the principle of virtual work, and it includes several earlier variational principles as special cases; the generation of the initial conditions is the novelty of this unified principle.

3- GEOMETRY OF LAMINAE

In the space E, consider a piezoelectric laminae composed of two perfectly conducting lower and face electrodes and (N-2) piezoelectric layers between them. Each constituent may possess distinct but uniform thickness $2h_m$, curvature and electromechanical properties. The regular region of laminae, V+S, with its boundary surface **S** is referred to the x^i-system of geodesic normal convected coordinates, the midsurface A of first layer $x^3=0$ being taken as the reference surface, such that the corresponding metric tensors of the undeformed laminae are given by

$$g_{\alpha\beta} = \mu^{\lambda}_{\alpha}\mu^{\sigma}_{\beta}\, a_{\lambda\sigma}\ , \qquad g^{\alpha\beta} = (\mu^{-1})^{\beta}_{\sigma}(\mu^{-1})^{\alpha}_{\lambda}a^{\lambda\sigma} \qquad (20)$$
$$g_{\alpha 3} = 0\ , \quad g_{33} = 1$$

with the shifters of the form

$$\mu^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - x^3 b^{\alpha}_{\beta}\ , \qquad \mu^{\alpha}_{\sigma}(\mu^{-1})^{\sigma}_{\beta} = \delta^{\alpha}_{\beta} \qquad (21)$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ denote the first and second fundamental forms of the reference surface A, and $c_{\alpha\beta}(=b_{\alpha\sigma}b^{\sigma}_{\beta})$ its third fundamental form. By use of the shifters, the components of a vector field, (X^i, X_i) and $(\bar{X}^i, \bar{X}_i)$, which are respectively referred to the base vectors of laminae space and those of reference surface are associated with one another as

$$X_\alpha = \mu^\beta_\alpha \bar{X}_\beta \;, \quad X^\alpha = (\mu^{-1})^\alpha_\beta \bar{X}^\beta \;; \quad \bar{X}_\alpha = (\mu^{-1})^\beta_\alpha X_\beta$$
$$\bar{X}^\alpha = \mu^\alpha_\beta X^\beta \;; \quad X_3 = X^3 = \bar{X}_3 = \bar{X}^3 \tag{22}$$

Besides, the equations of the form

$$x^3 = -h_1, \quad x^3 = 2h - h_1, \quad f(x^1, x^2) = 0 \tag{23}$$

define the lower and upper faces, S_{lf} and S_{uf}, and the edge boundary surface S_e of laminae. The reference surface A intersects the edge boundary surface along a Jordan curve C. The bonding surface between the m-th and (m+1)-th consituent is denoted by $A_{m,m+1}$, and the outward unit vector normal to S_e by ν_i and that to S_f by n_i. In addition to the x^i-system of local coordinates $x_m{}^i$ is introduced which is situated on the midsurface A_m of the m-th constituent, Thus, one reads

$$x^\alpha_m = x^\alpha, \quad x^3_m = x^3 - z_m \;; \; m=1,2,\ldots,N \tag{24}$$

where z_m is the distance between the surfaces A_m and A. Also, the equations as

$$x^3_m = 0 \;, \quad x^3 - z_m = 0 \tag{25}$$

clearly define the surface A and those by

$$x^3_m - h_m = 0, \quad x^3_{m+1} + h_{m+1} = 0 \tag{26}$$
$$x^3 - (z_m + h_m) = 0, \quad x^3 - (z_{m+1} - h_{m+1}) = 0$$

with

$$z_m = \sum_{r=1}^{m} (2 - \delta_{1r} - \delta_{mr}) h_r \tag{27}$$

the bonding surface $A_{m,m+1}$.

In the region of piezoelectric laminae,

$$2h/R_{min} << 1 \tag{28}$$

where R_{min} is the least principal radius of curvature of the midsurface A, and the elements of volume dV, of surface dS on S, of area dA on A and of line ds along C are given by

$$dV = \sqrt{g}dx^1dx^2dx^3 = dSdx^3 = \mu dAdx^3 \tag{29}$$

$$n_\alpha dS = \mu \upsilon_\alpha dsdx^3$$

with

$$\mu = |\mu^\alpha_\beta| = (g/a)^{1/2} = 1 - 2cx^3 + (x^3)^2 b$$

$$a = |a_{\alpha\beta}|, \quad b = |b^\alpha_\beta|, \quad c = 1/2\, b^\alpha_\alpha, \quad g = |g_{ij}| \tag{30}$$

where b and c denote the Gaussian and mean curvature of A, respectively; a more elaborate account of the results recorded can be found (e.g., [35]).

4- MECHANICAL DISPLACEMENTS, ELECTRIC POTENTIAL AND RESULTANTS FOR THE PIEZOELECTRIC STRAINED LAMINAE

In mathematical terms, the regular, finite and bounded region of piezoelectric laminae is defined by the fundamental assumption (28) which allows one to treat the laminae region as a two-dimensional medium. In addition to (29), all the field variables together with their derivatives are assumed to exist, to be single-valued and continuous functions of (x^i, t) in the closure of laminae region with no singularities of any kind, and not to vary widely across the thickness of layers. Accordingly, the fields of incremental mechanical displacements and electric potential which are chosen as a starting point of derivation are expressed by

$$\bar{u}^m_i = v^m_i + x^3 w^m_i \; ; \quad m = 1, 2, \ldots, N \tag{31}$$

and

$$\phi^m = \phi^m + x^3 \psi^m \; ; \quad m=2,3,\ldots,N-1 \qquad (32)$$

where $(v_i, w_i, \phi, \psi)^m$ are unknown a priori, independent and functions of x^α and t, only. In (31) v_α^m, v_3^m and w_α^m, and w_3^m represent, in this order, the extensional, flexural and thickness motions of the m-th constituent. Also, the use of (31) was shown to account for the coupled motions of laminae, as indicated already by Drumheller and Kalnins [39] and the author [7]. Moreover, in (32), m=1 and m=N are excluded, since the electrodes are perfectly conducting. When an alternative potential difference is applied to the electrodes, one reads

$$\phi^1 = \phi' \; , \; \phi^N = \phi'' \, , \quad \phi' = -\phi'' = \Phi_o \cos \omega t \qquad (33)$$

where Φ_o is a constant and ω the circular frequency.

In the piezoelectric laminae, the constituents are adhered one another and no relative deformation are permitted at their interfaces. Thus, the continuity of mechanical displacements and electric potential on, and that of tractions and surface charge across, the bonding interfaces $A_{m,m+1}$ are maintained. First, using (31), the continuity of incremental mechanical displacements is written as

$$v_i^m + (z_m + h_m)\, w_i^m = v_i^{m+1} + (z_{m+1} - h_{m+1})\, w_i^{m+1}$$

$$\text{no sum over } m; \; m=1,2,\ldots, N; \text{on } A_{m,m+1} XT \qquad (34)$$

This represents 3(N-1) constraints and reduces the number of the independent functions of displacements, 6N, in (31) to 3(N+1). The independent functions are chosen as

$$v_i^1, \; w_i^m \; ; \; m=1,2,\ldots,N \qquad (35)$$

and the rest of the displacement functions is expressed by

$$v_i^m = v_i^1 + \sum_{r=1}^{m} z_{rm} w_i^r \; ; \quad m= 2,3,\ldots,N \qquad (36)$$

with

$$z_{rm} = (2 - \delta_{1r} - \delta_{mr})\, h_{(r)} - \delta_{mr} z_{(m)} \qquad (37)$$

in terms of them.

Next, the continuity of electric potential is similarly expressed by

$$\phi^{m} + (z_m + h_m)\psi^{m} = \phi^{m+1} + (z_{m+1} - h_{m+1})\psi^{m+1} \tag{38}$$

no sum over m; m=2,3,..., N-1,

on $A_{m,m+1}XT$ (39)

and

$$\phi' = \phi^{2} + (z_2 - h_2)\psi^{2} \text{ on } A_{1,2}XT$$

$$\phi'' = \phi^{N-1} + (z_{N-1} + h_{N-1})\psi^{N-1} \text{ on } A_{N-1,N}XT \tag{40}$$

In view of the constraints (39), the (N-3) independent functions of electric potential are chosen as

$$\psi^{m} ;\ m=2,3,\ldots,N-2 \tag{41}$$

The dependent functions of electric potential are expressed by

$$\phi^{2} = \phi' - (z_2 - h_2)\psi^{2}$$

$$\phi^{m} = \phi' + \sum_{r=2}^{m} z_{rm}\psi^{r} ;\quad m= 2,3\ldots,N-2$$

$$\phi^{N-1} = -(z_{N-1}/h_{N-1})\,\phi'' + (1+z_{N-1}/h_{N-1})\sum_{r=2}^{N-2} h_r\psi^{r}$$

$$\psi^{N-1} = (1/h_{N-1})(\phi'' - \sum_{r=2}^{N-2} h_r\psi^{r}) \tag{42}$$

in terms of the independent functions (41).

Evidently, the linear representation (41) and (32) and the gradient equations (2) imply a distribution of mechanical strain for each constituent as

$$s_{ij} = \sum_{r=0}^{R} (x^3)^r \left[{}_r s_{ij}(x^\alpha, t)\right] \tag{43}$$

and that of electric field for each layer as

$$E_i = \sum_{r=0}^{R} (x^3)^r \left[{}_r E_i(x^\alpha, t)\right] \tag{44}$$

of which the explicit expressions are obtained in the next section.

Now, in accordance with the linear representation above, various field quantities are averaged over the thickness interval of each constituent for the subsequent development. Thus, the two-dimensional incremental resultants of stress are defined by

$$[N^{\alpha\beta},\ M^{\alpha\beta},\ K^{\alpha\beta}] = \int_Z [1,\ x^3,(x^3)^2]\, t^{\alpha\beta}\mu\, dx^3$$
$$[(Q^{\alpha},\ R^{\alpha});\ N^{33}] = \int_Z [(1,x^3)\, t^{\alpha 3};\ t^{33}]\,\mu dx^3 \tag{45}$$

and those of initial stress by

$$[N_o^{\alpha\beta},\ \ldots,\ R_o^{\alpha},\ N_o^{33}] = \int_Z [t_o^{\alpha\beta},\ \ldots, t_o^{33}]\,\mu dx^3 \tag{46}$$

those of acceleration by

$$A^i = \mu_o \ddot{v}^i + \mu_1 \ddot{w}^i\ ,\ B^i = \mu_1 \ddot{v}^i + \mu_2 \ddot{w}^i \tag{47}$$

with

$$\mu_n = I_n - 2cI_{n+1} + bI_{n+2} \tag{48}$$

where

$$I_n = \int_Z (x^3)^n dx^3 = [(z+h)^{n+1} - (z-h)^{n+1}]/(n+1)$$
$$n=0,1,\ldots, \tag{49}$$

those of traction by

$$(q^{\alpha},\ p^{\alpha}) = (\mu\mu_{\beta}^{\alpha} t^{3\beta}),\quad (q^3,\ p^3) = \mu t^{33}$$
$$\text{at } (x^3 = z+h,\ z-h) \tag{50}$$

and

$$(r_o^{\alpha},\ s_o^{\alpha}) = \{\mu t_o^{3\beta}[v^{\alpha}{}_{:\beta} - b_{\beta}^{\alpha} v_3 + x^3(w^{\alpha}{}_{:\beta} - b_{\beta}^{\alpha} w_3)]$$
$$+ \mu t_o^{33}\ w^{\alpha}\}\quad \text{at } (x^3 = z+h,\ z-h) \tag{51}$$

$$(r_o^3, s_o^3) = \{\mu t_o^{3\alpha}[v_{3,\alpha} + b_{\alpha}^{\beta} v_{\beta} + x^3(w_{3,\alpha} + b_{\alpha}^{\beta} w_{\beta})]$$
$$+ \mu t_o^{33}\ w_3\}\quad \text{at } (x^3 = z+h,\ z-h)$$

those of loads by

$$(N_{\star}^{\alpha},\ M_{\star}^{\alpha}) = \int_Z \tau_{\star}^{\beta} \mu_{\beta}^{\alpha}(1,\ x^3)\,\mu dx^3 \tag{52}$$

$$(N_*^3,\ M_*^3) = \int_Z \tau_*^3 (1,\ x^3)\mu dx^3$$

and

$$(S_*^\alpha,\ P_*^\alpha) = (\mu\mu_\beta^\alpha \tau_*^\beta) \text{ and } (S_*^3,\ P_*^3) = (\mu\tau_*^3)$$
$$\text{at } (x^3 = z+h,\ z-h) \tag{53}$$

and

$$l^i = q^i - p^i\ ,\ l_o^i = r_o^i - s_o^i \tag{54}$$
$$m^i = (z+h)q^i - (z-h)p^i;\ m_o^i = (z+h)r_o^i - (z-h)s_o^i$$

Besides, the two-dimensional resultants of electrical displacements in the form

$$(F^i,\ G^i) = \int_Z (1,\ x^3)\ D^i \mu dx^3 \tag{55}$$

those of surface charge by

$$(d,\ f) = (\mu D^3) \text{ at } (x^3 = z+h,\ z-h) \tag{56}$$

and

$$D = (z + h)d, \quad F = (z - h)e \tag{57}$$

and those of edge-surface charge by

$$(F,\ G) = \int_Z (1,\ x^3)\sigma\mu dx^3 \tag{58}$$

are introduced. In (45) - (58), the resultants of stress, initial stress and electric displacements are measured per unit length of the coordinate curves on A, those of acceleration, surface load and surface charge per unit area of A, and those of edge-load and edge-surface charge per unit length of C. Moreover, in terms of the foregoing definitions, the continuity of tractions and that of surface charge by

$$(q^i + r_o^i)^m - (p^i + s_o^i)^{m+1} = 0\ ;\ d^m - f^{m+1} = 0$$
$$\text{on } A_{m,m+1} X T \tag{59}$$

are given; the resultants can be similarly referred to the A_m of each constituent in place of A, [35].

5- DISTRIBUTIONS OF STRAIN AND ELECTRIC FIELD MACROSCOPIC CONSTITUTIVE EQUATIONS

The components of incremental strain of order (r) are obtained by use of the appropriate term of the unified principle (13), namely,

$$\delta J^{13}_{13} = \int_T dt \int_A \sum_{r=1}^{N} \{\int_Z [s_{ij} - 1/2(u_{i;j} + u_{j;i})] \delta t^{ij}\}^{(r)} \mu dA dx^3 = 0 \tag{60}$$

By inserting (31) into this equation, and then performing the integrals over the entire thickness of piezoelectric laminae, recalling the resultants of stress (45), one finally obtains the distribution of incremental strain in a variational form as

$$\begin{aligned}\delta J^{13}_{13} = \int_T dt \int_A \sum_{r=1}^{N} [&({}_0s_{\alpha\beta} - e_{\alpha\beta})\delta N^{\alpha\beta} \\ &+ ({}_1s_{\alpha\beta} - \varepsilon_{\alpha\beta})\delta M^{\alpha\beta} + ({}_2s_{\alpha\beta} - \gamma_{\alpha\beta})\delta K^{\alpha\beta} \\ &+ ({}_0s_{\alpha 3} - e_{\alpha 3})\delta Q^{\alpha} + ({}_1s_{\alpha 3} - \varepsilon_{\alpha 3})\delta R^{\alpha} \\ &+ ({}_0s_{33} - e_{33})\delta N^{33}]^{(r)} dA = 0\end{aligned} \tag{61}$$

This equation leads, as its Euler-Lagrange equations, to

$${}_0s_{ij} = e_{ij}, \quad {}_1s_{ij} = \varepsilon_{ij}, \quad {}_2s_{ij} = \gamma_{ij} \tag{62a}$$

where

$$\begin{aligned}e'_{\alpha\beta} &= 1/2(v_{\alpha:\beta} + v_{\beta:\alpha} - 2b_{\alpha\beta}v_3)' \\ e^{(m)}_{\alpha\beta} &= e'_{\alpha\beta} + 1/2\sum_{r=1}^{m} z_{rm}(w_{\alpha:\beta} + w_{\beta:\alpha} - 2b_{\alpha\beta}w_3)^{(r)} \\ e'_{\alpha 3} &= 1/2(v_{3,\alpha} + b^{\sigma}_{\alpha}v_{\sigma} + w_{\alpha})' \quad ; \ m=2,3,\ldots,N \\ e^{(m)}_{\alpha 3} &= e'_{\alpha 3} + 1/2\ [w^{(m)}_{\alpha} + \sum_{r=1}^{m} z_{rm}(w_{3,\alpha} + b^{\sigma}_{\alpha}w_{\sigma})^{(r)}] \\ e^{(m)}_{33} &= w^{(m)}_3 \quad ; \ m=1,2,\ldots,N\end{aligned} \tag{62b}$$

$$\begin{aligned}\varepsilon'_{\alpha\beta} = 1/2\ (&- b^{\sigma}_{\alpha}v_{\sigma:\beta} - b^{\sigma}_{\beta}v_{\sigma:\alpha} + 2c_{\alpha\beta}v_3 + w_{\alpha:\beta} \\ &+ w_{\beta:\alpha} - 2c_{\alpha\beta}w_3)'\end{aligned}$$

$$\varepsilon^{(m)}_{\alpha\beta} = 1/2\Big[(-b^{\sigma}_{\alpha}v_{\sigma:\beta} - b^{\sigma}_{\beta}v_{\sigma:\alpha} + 2c_{\alpha\beta}v_3)'$$

$$+(w_{\alpha:\beta} + w_{\beta:\alpha} - 2c_{\alpha\beta}w_3)^{(m)}$$

$$+\sum_{r=1}^{m} z_{rm}(-b^{\sigma}_{\alpha}w_{\sigma:\beta} - b^{\alpha}_{\beta}w_{\sigma:\beta} + 2c_{\alpha\beta}w_3)^{(r)}\Big] \tag{62c}$$

$$\varepsilon^{(m)}_{\alpha 3} = 1/2\ w_{3,\alpha};\quad \varepsilon^{(m)}_{33} = 0\ ;\quad m=1,2,\ldots,N$$

$$\gamma^{(m)}_{\alpha\beta} = 1/2\ (-b^{\sigma}_{\alpha}w_{\sigma:\beta} - b^{\sigma}_{\beta}w_{\sigma:\alpha} + 2c_{\alpha\beta}w_3)^{(m)}$$

$$\gamma^{(m)}_{3} = \gamma^{(m)}_{33} = 0\ ;\quad m= 1,2,\ldots,N$$

In deriving (62), the covariant derivatives of the dislacement vector are expressed with respect to surface metrics by means of the identities as

$$u_{\alpha;\beta} = \mu^{\sigma}_{\alpha}(\bar{u}_{\sigma:\beta} - b_{\sigma\beta}\bar{u}_3)\ ,\ldots,\ u_{3;3} = \bar{u}_{3,3} \tag{63}$$

Here, an overbar indicates the displacement components, as defined in (22)

In a similar manner, the distribution of electric field is found by use of the part of (13) in the form

$$\delta J^{23}_{23} = \int_T dt \int_A - \sum_{r=2}^{N-1}\{\int_Z[\delta D^i(E_i + \Phi_{,i})]\}^{(r)}\mu\, dA dx^3 \tag{64}$$

This yields the distribution of electric field in a variational form by

$$\delta J^{23}_{23} = \int_T dt \int_A \sum_{r=2}^{N-1}\Big[({}_oE_i - e_i)\delta F^i$$

$$+ ({}_1E_i - \varepsilon_i)\delta G^i\Big]^{(r)} dA = 0 \tag{65}$$

and

$${}_oE_i = e_i\ ,\ {}_1E_i = \varepsilon_i \tag{66}$$

Here, the denotations by

$$e^{(2)}_{\alpha} = (z_2 - h_2)\,\psi^{(2)}_{,\alpha}$$

$$e^{(m)}_{\alpha} = -\sum_{r=1}^{m} z_{rm}\,\psi^{(r)}_{,\alpha}\ ;\quad m=3,4,\ldots, N-2 \tag{67}$$

$$e_{\alpha}^{(N-1)} = -(1 + z_{N-1}/h_{N-1})\sum_{r=2}^{N=2} h_r \psi_{,\alpha}^{(r)}$$

$$e_3^{(m)} = \psi^{(m)}; \quad m=2,3,\ldots,N-2$$

$$e_3^{(N-1)} = 1/h_{N-1}(\phi'' - \sum_{r=2}^{N-2} h_r \psi^{(r)}) \qquad (67)$$

$$\varepsilon_{\alpha}^{(m)} = -\psi_{,\alpha}^{(m)}; \quad m=2,3,\ldots,N-2$$

$$\varepsilon_{\alpha}^{(N-1} = 1/h_{N-1}\sum_{r=2}^{N} h_r \psi_{,\alpha}^{(r)}$$

$$e_i' = e_i'' = \varepsilon_i' = \varepsilon_i'' = \varepsilon_3^{(m)} = 0; \quad m=2,3,\ldots,N$$

are introduced.

Now, the distributions (62) and (66) are substituted into the constitutive part of (13), and then the integrations are carried out with respect to the thickness coordinate and the resultants of stress and electric displacements are used wherever feasible, with the results in variational form by

$$\begin{aligned}\delta J_{12}^{12} = \int_T dt \int_A \sum_{r=1}^{N} \Big[&(N^{\alpha\beta} - N_C^{\alpha\beta})\delta e_{\alpha\beta} + (M^{\alpha\beta} - M_C^{\alpha\beta})\delta\varepsilon_{\alpha\beta} \\ &+ (K^{\alpha\beta} - K_C^{\alpha\beta})\delta\gamma_{\alpha\beta} + (Q^{\alpha} - Q_C^{\alpha})\delta e_{\alpha 3} + (R^{\alpha} - R_C^{\alpha})\delta\varepsilon_{\alpha 3} \\ &+ (N^{33} - N_C^{33})\delta e_{33}\Big]^{(r)} dA = 0\end{aligned} \qquad (68)$$

and

$$\begin{aligned}\delta J_{22}^{22} = \int_T dt \int_A \sum_{r=1}^{N} \Big[&(F^i - F_C^i)\delta e_i \\ &+ (G^i - G_C^i)\,\delta\varepsilon_i\Big]^{(r)} dA = 0\end{aligned} \qquad (69)$$

The Euler-Lagrange equations of (68) and (69) are the macroscopic constitutive relations in the form

$$\begin{aligned}&N^{\alpha\beta} - N_C^{\alpha\beta} = 0, \quad M^{\alpha\beta} - M_C^{\alpha\beta} = 0, \quad K^{\alpha\beta} - K_C^{\alpha\beta} = 0 \\ &Q^{\alpha} - Q_C^{\alpha} = 0, \quad R^{\alpha} - R_C^{\alpha} = 0, \quad N^{33} - N_C^{33} = 0\end{aligned} \qquad (70)$$

and

$$F^i - F_C^i = 0, \quad G^i - G_C^i = 0 \quad \text{on AXT} \qquad (71)$$

Here,

$$(N_c^{\alpha\beta}, M_c^{\alpha\beta}, K_c^{\alpha\beta}) = (\underset{\sim}{C}_o, \underset{\sim}{C}_1, \underset{\sim}{C}_2)^{\alpha\beta kl}(\underset{\sim}{s}_{kl})^T - (\underset{\sim}{C}_o, \underset{\sim}{C}_1, \underset{\sim}{C}_2)^{k\alpha\beta}(\underset{\sim}{E}_k)^T$$

$$(Q_c^{\alpha}, R_c^{\alpha}) = (\underset{\sim}{C}_o, \underset{\sim}{C}_1)^{\alpha 3kl}(\underset{\sim}{s}_{kl})^T - (\underset{\sim}{C}_o, \underset{\sim}{C}_1)^{k\alpha 3}(\underset{\sim}{E}_k)^T$$

$$N_c^{33} = C_o^{33kl}(\underset{\sim}{s}_{kl})^T - \underset{\sim}{C}_o^{k33}(\underset{\sim}{E}_k)^T \quad \text{on} \quad AXT \tag{72a}$$

and

$$(F_c^i, G_c^i) = (\underset{\sim}{C}_o, \underset{\sim}{C}_1)^{ijk}(\underset{\sim}{s}_{jk})^T + (\underset{\sim}{C}_o, \underset{\sim}{C}_1)^{ik}(\underset{\sim}{E}_k)^T \tag{72b}$$

In the above equations,

$$(\underset{\sim}{S}_{kl}) = (e_{kl}, \varepsilon_{kl}, \gamma_{kl}), \quad (\underset{\sim}{E}_k) = (e_k, \varepsilon_k, 0) \tag{73}$$

are defined. Also, the elastic stiffenesses by

$$\underset{\sim}{C}_n^{ij\ldots k} = (C_n, C_{n-1}, C_{n-2})^{ij\ldots k} \tag{74a}$$

with

$$C_n^{ij\ldots k} = \mu_n C^{ij\ldots k} \tag{74b}$$

are introduced.

6- GOVERNING EQUATIONS OF INCREMENTAL MOTION

In this section, within the order of approximation of the linear representation (31) and (32), the macroscopic stress equations of incremental motion, the macroscopic charge equations of electrostatics and the associated natural boundary and initial conditions are systematically derived. In the derivation, Mindlin's method of reduction is followed (see, e.g., [40], and the results are expressed in both variational and differential forms. Also, the governing equations of piezoelectric biased laminae are fully stated.

To begin with, the first term of (13) is written in the form

$$\delta J_{11}^{11} = \int_T dt \int_A \sum_{r=1}^{N} \int_Z [(t^{ij} + t_o^{ik} u^j{}_{;k})_{;i} - \rho a^j] \delta u_j)^{(r)} \mu \, dA dx^3 = 0 \tag{75}$$

for all the constituents of piezoelectric laminae. By substituting (31) into this variational integral, using various relations between space and surface tensors and their derivatives, performing integrations with respect to the thickness coordinate and recalling the resultants of stress, acceleration and load in Section 4, one finally arrives at the variational equation of incremental motion as

$$\delta J^{11}_{11} = \int_T dt \int_A \sum_{r=1}^{N} \Big[(V^i + U^i_o + l^i + l^i_o - \rho A^i)\delta v_i + (W^i + T^i_o + m^i + m^i_o - \rho B^i)\delta w_i \Big]^{(r)} dA = 0 \tag{76}$$

where

$$\begin{aligned} V^\alpha &= V^{\beta\alpha}{}_{:\beta} - b^\alpha_\sigma Q^\sigma, \quad V^3 = V^{\alpha 3}{}_{:\alpha} + b_{\alpha\beta}V^{\alpha\beta} \\ W^\alpha &= W^{\beta\alpha}{}_{:\beta} - Q^\alpha, \quad W^3 = W^{\alpha 3}{}_{:\alpha} - N^{33} + b_{\alpha\beta}W^{\alpha\beta} \end{aligned} \tag{77}$$

and

$$\begin{aligned} U^\alpha_o &= U^{\beta\alpha}_{o:\beta} - b^\alpha_\sigma U^{\sigma 3}_o, \quad U^3_o = U^{\alpha 3}_{o:\alpha} + b_{\alpha\beta}U^{\alpha\beta}_o \\ T^\alpha_o &= T^{\beta\alpha}_{o:\beta} - b^\alpha_\beta T^{\beta 3}_o - Q^\beta_o(v^\alpha{}_{:\beta} - b^\alpha_\beta v_3) \\ &\quad + (R^\beta_{o:\beta} - N^{33})w^\alpha \\ T^3_o &= T^{\alpha 3}_{o:\alpha} + b_{\alpha\beta}T^{\alpha\beta}_o - (R^\alpha_{o:\alpha} + N^{33}_o)w^3 \\ &\quad - Q^\alpha_o(v_{3,\alpha} + b^\beta_\alpha w_\beta) \end{aligned} \tag{78}$$

In these equations, the denotations of the form

$$\begin{aligned} V^{\alpha\beta} &= N^{\alpha\beta} - b^\beta_\sigma M^{\alpha\sigma}, \quad W^{\alpha\beta} = M^{\alpha\beta} - b^\beta_\sigma K^{\alpha\sigma} \\ V^{\alpha 3} &= Q^\alpha, \quad W^{\alpha 3} = R^\alpha \end{aligned} \tag{79a}$$

and

$$\begin{aligned} U^{\alpha\beta}_o &= N^{\alpha\sigma}(v^\beta{}_{:\sigma} - b^\beta_\sigma v_3) + M^{\alpha\sigma}_o(w^\beta{}_{:\sigma} - b^\beta_\sigma w_3) + Q^{\alpha 3}_o w^\beta \\ U^{\alpha 3}_o &= Q^{\alpha 3}_o w_3 + N^{\alpha\beta}_o(v_{3,\beta} + b^\sigma_\beta v_\sigma) + M^{\alpha\beta}_o(w_{3,\beta} + b^\sigma_\beta w_\sigma) \\ T^{\alpha\beta}_o &= M^{\alpha\sigma}_o(v^\beta{}_{:\sigma} - b^\beta_\sigma v_3) + K^{\alpha\sigma}_o(w^\beta{}_{:\sigma} - b^\beta_\sigma w_3) \\ T^{\alpha 3}_o &= M^{\alpha\beta}_o(v_{3,\beta} + b^\sigma_\beta v_\sigma) + K^{\alpha\beta}_o(w_{3,\beta} + b^\sigma_\beta w_\sigma) \end{aligned} \tag{79b}$$

are introduced in terms of (45)-(59). As a last step, using (36) and considering the continuity conditions of tractions (59), the macroscopic stress equation of incremental motion is expressed in variational form by

$$\delta J^{11}_{11} = \int_T dt \int_A \{ [\sum_{r=1}^{N} (\Gamma^i)^{(r)} + b^{i''} - c^{i'}] \delta v_i'$$
$$+ [\Pi^{i'} + h' \sum_{r=2}^{N} (\Gamma^i)^{(r)} + h'(c^{i'} + b^{i''})] \delta w^{i'}$$
$$\sum_{m=2}^{N-1} [(\Pi^i)^{(m)} + \sum_{r=m}^{N} z_{mr} (\Gamma^i)^{(r)} + 2h_m b^{i''}] \delta w_i^{(m)}$$
$$+ [\Pi^{i''} - (2h - h' - 2h'') \Gamma^{i''} + 2h'' b^{i''}] \delta w_i''\} dA = 0 \tag{80a}$$

with

$$\Gamma^i = V^i + U_o^i - \rho A^i, \quad \Pi^i = W^i + T_o^i - \rho B^i$$
$$b^i = q^i + r_o^i, \quad c^i = p^i + s_o^i \tag{80b}$$

in terms of the variations of independent displacement functions (31). From(80), the macroscopic stress equations folllow in differential from as

$$\sum_{r=1}^{N} (V^i + U_o^i)^{(r)} + b^{i''} - c^{i'} = \sum_{r=1}^{N} (\rho A^i)^{(r)}$$
$$(W^i + T_o^i)' + h' \sum_{r=2}^{N} (V^i + U_o^i)^{(r)} + h'(c' + b'')$$
$$= (\rho B^i)' + h' \sum_{r=2}^{N} (\rho A^i)^{(r)} \quad \text{on} \quad AXT$$
$$(W^i + T_o^i)^{(m)} + \sum_{r=m}^{N} z_{mr} (V^i + U_o^i)^{(r)} + 2h_m b^{i''}$$
$$= (\rho B^i)^{(m)} + \sum_{r=m}^{N} z_{mr} (\rho A^i)^{(r)}$$
$$(W^i + T_o^i)'' - (2h - h' - 2h'')(V^i + U_o^i)'' + 2h'' b''$$
$$= (\rho B^i)'' - (2h - h' - 2h'')(\rho A^i)'' \tag{81}$$

This equation or (80) represents 3(N+1) equations for the piezoelectric biased laminae. In deriving (80) and (81), details of lengthy computations are omitted; they are, however, given in a recent report [35].

In a similar manner, to derive the macroscopic charge equations of electrostatics, from (13), the variational volume integral of the form

$$\delta J^{21}_{21} = -\int_T dt \int_A \sum_{r=2}^{N} [\int_Z D^i_{;i} \delta\phi]^{(r)} \mu dA dx^3 = 0 \tag{82}$$

is evaluated for all the layers of piezoelectric laminae. In doing so, (42) is inserted into this equation and the integrations are carried out with respect to the thickness coordinate, and then the variational integral (82) is expressed by

$$\delta J^{21}_{21} = \int_T dt \int_A \sum_{r=1}^{N} \Big[(F^{\alpha}_{:\alpha} + d - f)\,\delta\phi + (G^{\alpha}_{:\alpha} - F^{3} + D - F)\,\delta\psi \Big]^{(r)} dA = 0 \tag{83}$$

with $\delta\psi' = \delta\psi'' = 0$. This equation is now written with respect to the variations of independent functions of electric potential (41) as

$$\begin{aligned} \delta J^{21}_{21} = \int_T dt \int_A \Big\{ & \sum_{m=2}^{N-3} \Big[\chi^{\alpha}_{m\,:\alpha} + \chi^{3}_{m} + \sum_{r=m}^{N-1} x_{mr} F^{\alpha}_{r:\alpha} \Big] \delta\psi^{(m)} \\ & + \Big[\chi^{\alpha}_{N-2\,:\alpha} - (z_{N-2} - h_{N-2}) F^{(N-2)\alpha}_{\;:\alpha} \\ & + (1 + z_{N-1}/h_{N-1}) h_{N-2} F^{\alpha}_{N-1\,:\alpha} \\ & + \chi^{3}_{N-2} \Big] \delta\psi^{(N-2)} \Big\} dA = 0 \end{aligned} \tag{84a}$$

with the denotations by

$$x_{mr} = \big[2 - \delta_{mr} + (-1 + z_{N-1}/h_{N-1})\,\delta_{N-1,r} \big] h_m - \delta_{mr} z_m$$

$$\chi^{\alpha}_{m} = G^{\alpha}_{m} - h_m/h_{N-1} G^{\alpha}_{N-1}, \quad \chi^{3}_{m} = -F^{3}_{m} + h_m/h_{N-1} F^{3}_{N-1} \tag{84b}$$

where the continuity of surface charge (59) is considered. The Euler-Lagrange equations of (84) are readily written as

$$\begin{aligned} & \chi^{\alpha}_{m:\alpha} + \chi^{3}_{m} + \sum_{r=m}^{N-1} x_{mr} F^{\alpha}_{r\,:\alpha} = 0 \;; \quad m = 2,3,\ldots,N-3 \\ & \chi^{\alpha}_{N-2\,:\alpha} - (z_{N-2} - h_{N-2}) F^{\alpha}_{N-2\,:\alpha} + \chi^{3}_{N-2} \\ & \quad + (1 + z_{N-1}/h_{N-1}) h_{N-2} F^{\alpha}_{N-1\,:\alpha} = 0 \quad \text{on} \quad A \times T \end{aligned} \tag{85}$$

which represent (N-3) equations. Thus, the macroscopic charge equations of electrostatics are expressed by (84) in variational form and by (85) in differential form.

Paralleling the derivation of the macroscopic divergence equations (75) - (85), the mechanical and electrical, natural boundary conditions of piezoelectric biased laminae are obtained by use of the variational principle (13). The tractions of biased laminae are prescribed on a part $S_t(=C_t XT)$ of S_e and S_{1f} and the surface charges on only S_e. To begin with, consider the pertinent term of (13) for the mechanical boundary conditions, namely,

$$\delta I^{11}_{11} = \int_T dt \oint_{C_t} \{ \sum_{r=1}^{N} \int_Z [\tau^j_* - \upsilon_\alpha (t^{\alpha j} + t_o^{\alpha k} u^j_{;k})] \delta u_j \}^{(r)} \mu \, ds dx^3$$
$$+ \int_T \int_{S_{1f}} \{ [\tau^j_* - n_3 (t^{3j} + t_o^{3k} u^j_{;k})] \delta u_j \}' \mu \, dA = 0 \qquad (86)$$

After evaluation as before, this equation leads to the natural boundary conditions of tractions in variational form as

$$\delta I^{11}_{11} = \int_T dt \oint_{C_t} \{ \sum_{r=1}^{N} [N^j_* - \upsilon_\alpha (V^{\alpha j} + U_o^{\alpha j})]^{(r)} \delta v'_j$$
$$+ \{ [M^j_* - \upsilon_\alpha (W^{\alpha j} + T_o^{\alpha j})]' + h' \sum_{r=2}^{N} [N^j_* - \upsilon_\alpha (V^{\alpha j}$$
$$+ U_o^{\alpha j})]^{(r)} \} \delta w'_j + \sum_{m=1}^{N-1} \{ [M^j_* - \upsilon_\alpha (W^{\alpha j} + T_o^{\alpha j})]^{(m)}$$
$$+ \sum_{r=m}^{N} z_{mr} [N^j_* - \upsilon_\alpha (V^{\alpha j} + U_o^{\alpha j})]^{(r)} \} \delta w_j^{(m)} + \{ [M^j_*$$
$$- \upsilon_\alpha (W^{\alpha j} + T_o^{\alpha j})]'' - (2h - h' - h'') [N^j_*$$
$$- \upsilon_\alpha (V^{\alpha j} + U_o^{\alpha j})]'' \} \delta w''_j \} ds \qquad (87)$$
$$+ \int_T dt \int_{S_{1f}} [(P^{j\prime}_* - c^{j\prime}) \delta v'_j + h' (P^j_* - c^{j\prime}) \delta w'_j] dA = 0$$

which yields readily the boundary conditions of tractions as follows

$$\sum_{r=1}^{N} [N^j_* - \upsilon_\alpha (V^{\alpha j} + U_o^{\alpha j})]^{(r)} = 0 \qquad (88a)$$

$$[M^j_* - \upsilon_\alpha (W^{\alpha j} + T_o^{\alpha j})]' + h_1 \sum_{r=2}^{N} [N^j_* - \upsilon_\alpha (V^{\alpha j} + U_o^{\alpha j})]^{(r)} = 0;$$

$$[M^j_* - \upsilon_\alpha (W^{\alpha j} + T_o^{\alpha j})]^{(m)} \qquad m=2,3,\ldots,N-1$$

$$+\sum_{r=m}^{N} z_{mr}\left[N_{\star}^{j} - \upsilon_{\alpha}(V^{\alpha j} + U_{a}^{\alpha j})\right]^{(r)} = 0$$

$$\left[M_{\star}^{j} - \upsilon_{\alpha}(W^{\alpha j} + T_{o}^{\alpha j})\right]'' \qquad \text{on } C_{t}XT \qquad (88a)$$

$$-(2h - h' - 2h'')\left[N_{\star}^{j} - \upsilon_{\alpha}(V^{\alpha j} + U_{o}^{\alpha j})\right]'' = 0$$

and

$$P_{\star}^{j} - (p^{j} + s_{o}^{j})' = 0 \quad \text{on } S_{1f}XT \qquad (88b)$$

in differential form as well. Besides, the natural boundary conditions of mechanical displacements by

$$v_{i}' - v_{i}^{\star\prime} = 0\ ,\ w_{i}^{m} - w_{i\star}^{m} = 0 \quad \text{on } C_{u}XT$$

$$v_{i}'' - \delta_{i}^{\star''} = 0 \quad w_{i}'' - \gamma_{i}^{\star''} = 0 \quad \text{on } S_{uf}XT \qquad (89)$$

are recorded.

In like manner, substituting (40) into the variational surface integral of the form

$$\delta I_{21}^{21} = \int_{T} dt \oint_{C} \sum_{r=2}^{N-1} \left[\int_{Z} (\upsilon_{\alpha} D^{\alpha} - \sigma_{\star})(\delta\phi + x^{3}\delta\psi)\right]^{(r)} \mu\, ds\, dx^{3} = 0 \qquad (90)$$

and evaluating it, one reads

$$\delta I_{21}^{21} = \int_{T} dt \oint_{C} \sum_{r=2}^{N-1} \left[(\upsilon_{\alpha} F^{\alpha} - F_{\star})\delta\phi + (\upsilon_{\alpha} G^{\alpha} - G_{\star})\delta\psi\right]^{(r)} ds = 0 \qquad (91)$$

in terms of the resultants (55) - (58). By use of (42), the natural boundary conditions of surface charge are expressed in variational form by

$$\delta I_{21}^{21} = \int_{T} dt \oint_{C} \Big\langle \sum_{m=2}^{N-3} \left[H_{\star}^{(m)} - \upsilon_{\alpha}(\chi_{m}^{\alpha} + \sum_{r=m}^{N-1} \chi_{mr} F_{r}^{\alpha})\right]\delta\phi^{m}$$
$$+ \{H_{\star}^{N-2} - \upsilon_{\alpha}\left[\chi_{N-2}^{\alpha} + (z_{N-2} - h_{N-2})F_{N-2}^{\alpha}\right.$$
$$\left.+ (1 + z_{N-1}h_{N-1})h_{N-2}F_{N-1}^{\alpha}\right]\}\delta\psi^{(N-1)}\Big\rangle ds = 0 \qquad (92)$$

and those in differential form by

$$H_{\star}^{(m)} - \upsilon_{\alpha}(\chi_{m}^{\alpha} + \sum_{r=m}^{N-1} \chi_{mr} F_{r}^{\alpha}) = 0\ ; \quad m = 2, 3, \ldots, N-3 \qquad (93)$$

$$H_*^{N-2} - \nu_\alpha [\chi^\alpha_{N-2} + (z_{N-2} - h_{N-2}) F^\alpha_{N-2} + (1 + z_{N-1}/H_{N-1}) h_{N-2} F^\alpha_{N-1}] = 0 \quad \text{along CXT} \tag{93}$$

Here,

$$H_*^m = G_*^m - (h_m/h_{N-1}) G_*^{N-1} + \sum_{r=m}^{N-1} x_{mr} F_*^r \; ; \; m=2,3,\ldots N-3 \tag{94}$$

$$H_*^{N-2} = G_*^{N-2} - (h_{N-2}/h_{N-1}) G_*^{N-1} - (z_{N-2} - h_{N-2}) F_*^{N-2} + (1 + z_{N-1}/h_{N-1}) h_{N-2} F_*^{N-1} \tag{95}$$

with

$$(F_*, G_*)^m = \int_{Z_m} (1, x^3) \sigma_*^{(m)} \mu dx^3 \tag{96}$$

are introduced.

Lastly, an evaluation of the volume integrals I_i^i of (13) yields the natural initial conditions of mechanical displacements and electric potential by

$$v_i^{!}(x^\alpha, t_o) - \alpha_i^{*'}(x^\alpha) = 0, \; \dot{v}_i^{!}(x^\alpha, t_o) - \beta_i^{*'}(x^\alpha) = 0$$

$$w_i^m(x^\alpha, t_o) - \xi_i^{*m}(x^\alpha) = 0, \; \dot{w}_i^m(x^\alpha, t_o) - \xi_i^{*m}(x^\alpha) = 0 \quad m=1,2,\ldots, N \tag{97}$$

$$\psi^m(x^\alpha, t_o) - \eta_i^{*m}(x^\alpha) = 0; \quad m=2,3,\ldots, N-1 \quad \text{on } A(t_o) \tag{98}$$

where $\alpha_i^* \ldots \eta_i^*$ are given functions.

Up to now, the set of two-dimensional, approximate equations of piezoelectric biased laminae is systematically and consistently derived by means of the unified variational principle (13) together with the linear representation (31) and (32). The electroelastic equations are given both in variational and differential forms at the perturbed state. Similarly the governing equations can be derived at the unperturbed state for the static behavior of laminae at the unperturbed state. This is recorded in [35].

7- A DIRECT METHOD OF SOLUTION

In this section a general algorithm is pointed out which is based upon Kantorovich's method for the fields of mechanical displacements and electric potential, as an alternative of the macromechanical analysis of piezoelectric biased laminae. The algorithm starts from the integral principle of (3) in lieu of its Euler-Lagrange differential equations and it rests entirely upon a selection of the fields for each constituent under the ad hoc assumptions for the piezoelectric region in Section 4. The method can be readily and successfully employed by means of high-speed digital computers for the macromechanical analysis.

To begin with, the fields of incremental mechanical displacements and electric potential are expressed by

$$\bar{u}_i^m(x^j,t) = \sum_{p+q+r=0}^{R} \left[f_i^{pqr}(x^\alpha,t)\; f_r(x^3)\right]^m$$
$$\phi^m(x^i,t) = \sum_{p+q+r=0}^{R} \left[g_{pqr}(x^\alpha,t)\; g_r(x^3)\right]^m \qquad (99a)$$

with

$$\left[f_i^{pqr}\,,\; g_{pqr}\right]^m = \left[\alpha_{pqr}(t)\,u_i^{pq}(x^\alpha)\,,\,\beta_{pqr}(t)\,\phi_{pq}(x^\alpha)\right]^m$$

$$\left[f_r(x^3),\; g_r(x^3)\right] = (x^3)^r \qquad (99b)$$

Here, (α_{pqr} and β_{pqr}) are the functions to be determined, whereas (u_i^{pq} , ϕ_{pq}) are the approximating functions to be chosen appropriately in order to satisfy all or some of the given boundary conditions; the rest of constraint conditions can be taken into account through Lagrance multipliers as illustrated by the author [35]. The approximating functions should be selected as simple as possible so that operations involving them can be carried out easily.

With the help of (99), the evaluation of the variational integral (3) leads readily to a system of ordinary differential equations in terms of $\alpha_{pqr}(t)$ and $\beta_{pqr}(t)$. The system of equations can be reduced to that of nonlinear algebraic equations for the case when vibrations and wave propagation are considered in the piezoelectric biased laminae.

The results with some applications are presented in detail in a forthcoming report [35].

8- ON SPECIAL CASES

Various cases involving special geometry, material and incremental motion of piezoelectric laminae may be readily investigated with the help of the general results derived in invariant form in the previous sections. Here attention is first limited to the case of piezoelectric plates in which the curvature effect vanishes, namely, $b^{\alpha}_{\beta}=\mu^{\alpha}_{\beta}=0$ (cf., [24]). The results for one layer (N=1) agree with those [26]. A complete linearization in the results leads to the linear theory of piezoelectric laminae. In such a case, it is shown by logarithmic convexity argument that the conditions (87) - (98) are sufficient to ensure the uniqueness in solutions of the electroelastic equations of laminae. This and a variety of applications of the general results to particular problems are given in a recent report [35]. Further, special classes of materials for the constituents of piezoelectric laminae may be considered in the macroscopic constitutive relations (68)-(73), and also special kinematics may be introduced in (31), (32) and (99).

9- SUMMARY AND CONCLUSIONS

Established herein is a systematic and consistent derivation of the two-dimensional electroelastic equations of piezoelectric laminae under a mechanical bias by means of the unified variational principle (13) together with the linear representation (31) and (32). The electroelastic equations given in both differential and variational forms govern all the incremental types of laminae motions. The variational principle generates all the fundamental equations of piezoelectric strained media. The results contain some of earlier results as special cases [35]. Lastly, an extension of the present results to viscoelastic and electromagnetic layers will be reported elsewhere.

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CHAPTER 7

NUMERICAL ALGORITHMS FOR DYNAMICS OF PIEZOELECTRIC LAMINAE

ABSTRACT

Numerical algorithms are developed for the macromechanical analysis of dynamic response of a piezoelectric strained laminae. The laminae which is under a general state of mechanical, static bias may comprise any number of bonded layers, each with a distinct but uniform thickness, curvature and electromechanical properties. In the first part of the paper, a direct method of solution based essentially on Kantorovich's method is presented for the macromechanical analysis. The effects of elastic stiffenesses of, and the interactions between, layer of the laminae are all taken into account and all the continuity conditions are maintained at the interfaces of layers. The resulting equations accommodate the extensional, thickness shear and flexural as well as coupled incremental motions of the laminae. In the second part, the governing equations of piezoelectric strained laminae are recorded and then the method of moments is described, as an alternative, for the macromechanical analysis. In the third part, special cases involving the geometry, motion and material properties of piezoelectric strained laminae are indicated.

1- INTRODUCTION

Piezoelectric laminae and/or composite elements with their desirable vibration characteristics for ultrasonic applications are of recent demand in different technologies. The use of these elements, the basic ideas underlying their sum and product properties and the mathematical models to describe their dynamic response are elaborated [1-3]. The research is still quite active on the design, and in determining the dynamic charasteristics, of laminae elements. Basically, two types of mathematical models exist for the dynamic analysis of these elements: the effective modulus model and the effective stiffeness model. The effective modulus model which replaces an element by a representative homogeneous medium is relatively simple, but it abrogates the mutual coupling of layers, and it is generally suitable for the

static analysis of laminae. The effective stiffeness model which incorporates all the essential features of layers accounts for the static as well as dynamic response of laminae. On the basis of these models, investigations were recorded to be abundant for certain vibrational modes of piezoelectric laminae with polarization in different directions, whereas to be rather scanty for derivation of the electroelastic relations of laminae which accommodate all the types of vibrational modes [3].

It was Mindlin [4] who first described to deduce, in a consistent and systematic manner, lower order equations from the three-dimensional equations of elastodynamics. By use of Mindlin's method of reduction, the author [5] obtained a system of two-dimensional, approximate governing equations for vibrations of thermopiezoelectric laminae. Karnaukhov and Kirichok [6,7] set up the governing equations of laminated piezoelectric shells by taking into account the effect of viscosity and temperature. Also, Evseichik, Rudnitskii and Shul'ga [8] derived the electroelastic layers. More recently, the author [9] extended his works [5,10] so as to derive the macroscopic equations of a piezoelectric laminae under a bias. Mikhailov and Parton [11] reported analytical studies involving certain vibrations of multilayered piezoelectric elements subjected to polarizations in different directions and various electric and/or mechanical loading conditions using standard approximate methods of numerical analysis. In these studies, no results were provided yet for the existence and uniqueness of solutions (cf., [5] for the uniqueness only), and also a unified algorithm for numerical solutions is still unavailable; the latter is taken up in this paper.

Various methods were used for numerical solutions of the initial-mixed boundary value problems defined by the governing differential equations of one and two-dimensional piezoelements [2,3] . Of the methods, the method of Green's potential function [12] , the Ritz-Galerkin method [13] , the asymptotic method [14] , the finite difference method [15] , the method of least squares [16] , the Fourier expansion collocation method [17] , the method of Laplace transform [18] , the method of fast Fourier transform [19], the method of z-transform [20] , the finite element method [21,22] and the boundary element method [23] were mentioned. Numerous treatises appeared on the applications of these methods in different branches of electrical engineering. Noteworthy is a first textbook by Silvester and Ferrari [24] on the finite element applications and the treatises by Harrington [25] and Wang [26] on the field computation by the method of moments. The method of moments with desirable features in

electromagnetics may form a universal approach to the macromechanical analysis of piezoelectric elements, though it has no application yet. This method is now described for the analysis of dynamic response of a piezoelectric laminae. Besides, a direct method of solution which is essentially based on Kantorovich's method |27| is presented, as an alternative, for the macromechanical analysis.

Briefly stated, the notation to be used herein is given in the remaining of this section. The next section contains a summary of the fundamental equations of piezoelectric medium under a mechanical bias; they are recorded in both differential and variational forms. Section 3 deals with the geometry of a piezoelectric laminae, and also, for ease of quick reference, the relations between space and surface tensor are recorded in this section. The strained laminae may comprise any number of bonded layers, each with a distinct but uniform thickness, curvature and electromechanical properties. In Section 4, a direct method of solutions which is based on Kantorovich's method is described for the macromechanical analysis of dynamic response of the piezoelectric strained laminae. In addition, the method of moments is developed as an alternative of the macromechanical analysis in Section 5. Special cases involving the geometry, motion and material properties of piezoelectric strained laminae are presented in Section 6 and concluding remarks in Section 7.

N o t a t i o n

In the paper, standard tensor notation is freely used in a Euclidean 3-Space E. Accordingly, Einstein's summation convention is implied over all repeated Latin indices (1,2,3) and Greek indices (1,2) that stand for space and surface tensors, respectively, unless they are put within parantheses. In the space E, the x^i-system is identified by a fixed, right-handed system of geodesic normal convected (intrinsic) coordinates. All the field quantities are described in Lagrangian formulation, they are indicated by a zero index in the initial state, by an asterisk when they are prescribed and by an overbar when they are referred to the base vectors of layer midsurface. A superposed dot stands for time differentiation, a comma for partial differentiation with respect to the indicated space coordinate, and a semicolon and a colon for covariant differentiation

with respect to the indicated coordinate, using the space and surface metrics, respectively. The index (m) which takes the values 1,2,...,N refers to the m-th constituent from the lower face or first layer of piezoelectric laminae, and also a prime is assigned to stand for the upper face of a layer and a double prime for its lower face. Moreover, B(t) refers to a regular, finite and bounded region B of the space E at time t, $\bar{B}(=B\cup\partial B)$ to the closure of B with its boundary surface ∂B, T to the time interval $[t_o,t_1)$, ζ to the thickness coordinate x^3, Z to the thickness interval $[z-h,z+h]$ with the layer thickness 2h and $\bar{B}$xT to the domain of definitions of the functions of the space coordinates and time.

2- FUNDAMENTAL EQUATIONS FOR INCREMENTAL MOTIONS IN PIEZOELECTRICITY

In the Euclidean 3-space E, let $B+\partial B$ with its boundary surface ∂B, denote a regular, finite and bounded region of piezoelectric elastic medium at its reference (initial) state at time $t=t_o$. The piezoelectric region which is subjected to static initial stresses is in equilibrium. This initial (unperturbed) state acquires its spatial (perturbed) state by small incremental motions superimposed upon the *finite static* deformation of the piezoelectric region at the time interval $T=[t_o,t_1)$. The elastic region is referred to by a fixed, right-handed system of general convected (intrinsic) coordinates x^i in the space E. The entire boundary surface ∂B consists of the complementary regular subsurfaces $(\partial B_t,\partial B_u)$ or $(\partial B_\sigma,\ \partial B_\phi)$, and the unit outward vector normal to ∂B^u is denoted by n_i. The domain of definitions for the functions (x^i,t) is denoted by $\bar{B}$xT, where $\bar{B}(=B\cup\partial B)$ being the closure of the region. All the field quantities of the piezoelectric region are described in Lagrangian formulation.

Now, the three-dimensional fundamental equations of incremental motions of the piezoelectric strained region are recorded [28,29] .

D i v e r g e n c e   e q u a t i o n s

$$\mathcal{L}^j=(t^{ij}+t_o^{ik}u^j{}_{;k})_{;i}-\rho a^j=0 \quad \text{in } \bar{B}XT \tag{1}$$

$$\mathcal{L}=D^i{}_{;i}=0 \quad \text{in } \bar{B}XT \tag{2}$$

with the definitions

t^{ij}, t_o^{ij}	initial and incremental stress tensors
u_i	incremental mechanical displacement vector
ρ	mass density of the undeformed body
a^i	Lagrangian acceleration vector $(=\ddot{u}^i)$
D^i	electric displacement vector

G r a d i e n t   e q u a t i o n s

$$K_{ij}=S_{ij}-\frac{1}{2}(u_{i;j}+u_{j;i})=0 \quad \text{in} \quad \bar{B}XT \tag{3}$$

$$M_i=E_i-(-\phi_{,i})=0 \quad \text{in} \quad \bar{B}XT \tag{4}$$

where

S_{ij}	strain tensor
E_i	quasi-static electric field vector
ϕ	electric potential

C o n s t i t u t i v e   r e l a t i o n s

$$L^{ij}=t^{ij}-(C^{ijkl}S_{kl}-C^{kij}E_k)=0 \quad \text{in} \quad \bar{B}XT \tag{5}$$

$$K^i=D^i-(C^{ijk}S_{jk}+C^{ij}E_j)=0 \quad \text{in} \quad \bar{B}XT \tag{6}$$

where

C^{ijkl}	elastic constants $(=C^{jikl}=C^{klij})$
C^{ijk}	piezoelectric strain constants $(=C^{ikj})$
C^{ij}	dielectric permittivity $(=C^{ji})$

B o u n d a r y   c o n d i t i o n s

$$L_\star^j=\tau_\star^j-n_i(t^{ij}+t_o^{ik}u^j_{;k})=0 \quad \text{on} \quad \partial B_t XT \tag{7}$$

$$K_i^\star=u_i^\star-u_i=0 \quad \text{on} \quad \partial B_u XT \tag{8}$$

$$L_\star=\sigma_\star-n_iD^i=0 \quad \text{on} \quad \partial B_\sigma XT \tag{9}$$

$$K_\star=\phi_\star-\phi=0 \quad \text{on} \quad \partial B_\phi XT \tag{10}$$

with

$$\tau^j=n_i\tau^{ij} \tag{11}$$

$$\tau^{ij} = t^{ij} + t_o^{ik} u^j_{;k} \tag{12}$$

where

τ^j Trefftz stress vector

σ surface charge

I n i t i a l c o n d i t i o n s

$$\begin{aligned} & u_i(x^j, t_o) - v_i^*(x^j) = 0 \\ & \dot{u}_i(x^j, t_o) - w_i^*(x^j) = 0 \qquad \text{in } B(t_o) \\ & \phi\ (x^i, t_o) - \psi_*(x^j) = 0 \end{aligned} \tag{13}$$

The differential governing equations (1)-(13) of incremental motions are alternatively stated in variational form [10,30,31] by use of the principle of virtual work (or Hamilton's principle). The principle of virtual work is stated for the piezoelectirc strained region as an assertation of the form.

$$-\delta\Sigma + \delta\Gamma + \delta^* W = 0$$

with the denotations

$$\delta\Sigma = \int_B (T^{ij}\delta S_{ij} - D^i \delta E_i)\, dV\ ,\quad \delta\Gamma = \frac{1}{2}\delta \int_B \rho \dot{u}^i \dot{u}_i\ dV,$$

$$\delta^* W = \int_{\partial B} (T_*^i \delta u_i + \sigma_* \delta\phi)\, dS \tag{14}$$

where $\delta^* W$ stands for the work done by external mechanical and electrical forces, and δ^* with an asterisk is used to distinguish it from the variation operator δ. In equation (14), the quantities of the form.

$$T^{ij} = t^{ij} + t_o^{ij}\ ,\quad S_{ij} = s_{ij} + \frac{1}{2} u^k{}_{;i} u_{k;j}\ ,\quad T^i = t^i + t_o^i \tag{15}$$

are introduced. Integrating equation (14) over the time interval T, carrying out variations, applying the Green-Gauss transformation of integrals for the regular region B and implying the usual arguments on incremental quantities, one finally arrives a two-field variational principle [10] as

$$\delta\chi\{u_i, \phi\} = \int_T dt \int_B (\mathcal{L}^i \delta u_i + \mathcal{L}\delta\phi)\, dV + \int_T dt \int_{\partial B} (\mathcal{L}_*^i \delta u_i + \mathcal{L}_* \delta\phi)\, dS = 0 \tag{16}$$

which yields the equations of incremental motion (1), the charge equation of electrostatics (2) and the associated natural boundary conditions of tractions and surface charge (7) and (9). The variational principle (16) includes the rest of the governing equations (3)-(6), (8),(10) and (13) together with

$$\delta u_i = \delta\phi = 0 \quad \text{in} \quad B(t_o) \quad \text{and} \quad B(t_1) \tag{17}$$

as its constraint conditions.

3- GEOMETRY OF LAMINAE

With reference to the x^i-system of general convected coordinates in the Euclidean space E, a thin piezoelectric strained laminae V+S, with its smooth boundary surface S is considered at its initial (unperturbed) state at time $t=t_o$ and it is brought into its spatial state through some elastic process at $T=[t_o, t_1)$. The laminae is composed of N constituents: two perfectly conducting, lower and upper face electrodes and (N-2) piezoelectric layers between them. The lower face electrode is indicated by a prime or m=1, the layers by m=2,...,(N-1), and the upper face electrode by a double prime or m=N. Each constituent may possess distinct but uniform thickness 2 h_m(m=1,2,...,N), curvature and electromechanical properties. The midsurface A of first layer $x^3=0$ is taken as the reference surface such that

$$x^3=-h_1=-h' \;,\; x^3=2H-h_1 \;,\; f(x^\alpha)=0 \tag{18}$$

define the lower and upper faces, S_{lf} and S_{uf}, and the edge boundary surface S_e of the laminae. The surface S_e is taken as a right cyclindrical surface whose generators lie along the normal to S_{lf} and S_{uf}, and it intersects them along closed, non-intersecting smooth Jordan curves C_m [32]. The bonding surface between the m-th and (m+1)th constituents is denoted by $A_{m,m+1}$, the midsurface of the m-th constituent by A_m and the unit outward vector normal to A' or A" by n_i and that to the edge boundary surface of the m-th constituent by ν_i.

On the reference surface A, $x^3=0$ is chosen positively upward and the x^α- coordinate curves form a system of curvilinear coordinates. In addition, a system of local coordinates x^i_m situated on A_m is introduced by

$$x_m^\alpha = x^\alpha \quad , \quad x_m^3 = x^3 - z_m, \quad m=1,2,\ldots,N \tag{19}$$

Here, z_m is the distance between the parallel midsurfaces A and A_m^m, hence, the parametric equations of the form

$$x_m^3 = 0 \quad , \quad x^3 - z_m = 0 \tag{20}$$

and

$$\begin{gathered} x_m^3 - h_m = 0 \quad , \quad x_{m+1}^3 + h_{m+1} = 0 \\ x^3 - (z_m + h_m) = 0 \quad , \quad x^3 - (z_{m+1} - h_{m+1}) = 0 \end{gathered} \tag{21}$$

with

$$z_m = \sum_{r=1}^{m} (2 - \delta_{1r} - \delta_{mr}) h_r \tag{22}$$

clearly define the midsurface A_m and the bonding surface $A_{m,m+1}$.

In the x^i-coordinate system, the position vector $\underset{\sim}{R}$ of a generic point P in the laminae space V takes the form

$$\underset{\sim}{R}(x^i) = \underset{\sim}{r}(x^\alpha) + x^3 \; \underset{\sim}{a}_3(x^\alpha) \tag{23}$$

with

$$\underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_3 = 0 \quad , \quad \underset{\sim}{a}_3 \cdot \underset{\sim}{a}_3 = 1 \tag{24}$$

Here, $\underset{\sim}{r}$ represents the position vector of the projection of P *on*, $\underset{\sim}{a}_\alpha = R_{,\alpha}(x^\beta, 0)$ *the* covariant base vectors of, and $\underset{\sim}{a}_3$ the unit vector normal to, the reference surface A. Thus, the base vectors,and metric and conjugate tensors of the space V are defined by

$$\underset{\sim}{g}_\alpha = \underset{\sim}{a}_\alpha + x^3 \underset{\sim}{a}_{3,\alpha} = \mu_\alpha^\beta \underset{\sim}{a}_\beta \; ; \; \underset{\sim}{g}^\alpha = (\overset{-1}{\mu})_\beta^\alpha \underset{\sim}{a}^\beta \quad ; \; \underset{\sim}{g}_3 = \underset{\sim}{g}^3 = \underset{\sim}{a}_3 = \underset{\sim}{a}^3 \tag{25}$$

and

$$g_{\alpha\beta} = \mu_\alpha^\sigma \mu_\beta^\nu a_{\sigma\nu} \quad , \quad g^{\alpha\beta} = (\overset{-1}{\mu})_\sigma^\alpha (\overset{-1}{\mu})_\nu^\beta \; a^{\sigma\nu}, \; g_{\alpha 3} = 0 \; , \; g_{33} = 1 \tag{26}$$

with the shifters of the form

$$\mu^{\alpha}_{\beta}=\delta^{\alpha}_{\beta}-b^{\alpha}_{\beta}x^{3},\quad \mu^{\alpha}_{\sigma}(\overset{-1}{\mu})^{\sigma}_{\beta}=\delta^{\alpha}_{\beta},\quad \mu(\overset{-1}{\mu})^{\alpha}_{\beta}=\delta^{\alpha}_{\beta}+(b^{\alpha}_{\beta}-b^{\sigma}_{\sigma}\delta^{\alpha}_{\beta}),$$

$$\mu=|\mu^{\alpha}_{\beta}| \tag{27}$$

and those of A by

$$a_{\alpha\beta}=\underset{\sim}{a}_{\alpha}\cdot\underset{\sim}{a}_{\beta}=g_{\alpha\beta}(x^{\sigma},0),\quad a^{\alpha\beta}=g^{\alpha\beta}(x^{\sigma},0),\quad a^{\alpha\sigma}a_{\beta\sigma}=\delta^{\alpha}_{\beta},$$

$$a_{\alpha 3}=a^{\alpha 3}=0,\quad a_{33}=a^{33}=1 \tag{28}$$

in which $a_{\alpha\beta}, b_{\alpha\beta}$, and $c_{\alpha\beta}=b_{\alpha\sigma}b^{\alpha}_{\beta}$ stand for the first, second, and third fundamental forms of A, respectively. By use of the shifters, the components of a vector field of the form

$$\underset{\sim}{\chi}=\chi^{i}\underset{\sim}{g}_{i}=\chi_{i}\underset{\sim}{g}^{i}=\bar{\chi}_{\alpha}\underset{\sim}{a}^{\alpha}+\bar{\chi}_{3}\underset{\sim}{a}^{3}=\bar{\chi}^{\alpha}\underset{\sim}{a}_{\alpha}+\bar{\chi}^{3}\underset{\sim}{a}_{3} \tag{29}$$

which are referred respectively to the base vectors of the laminae space V and those of the reference surface A are assaciated with one another as

$$\chi_{\alpha}=\mu^{\beta}_{\alpha}\bar{\chi}_{\beta},\quad \chi^{\alpha}=(\overset{-1}{\mu})^{\alpha}_{\beta}\bar{\chi}^{\beta};\quad \bar{\chi}^{\alpha}=\mu^{\alpha}_{\beta}\chi^{\beta},\quad \bar{\chi}_{\alpha}=(\overset{-1}{\mu})^{\beta}_{\alpha}\chi_{\beta};$$

$$\chi^{3}=\chi_{3}=\bar{\chi}^{3}=\bar{\chi}_{3} \tag{30}$$

In addition, the relations between space and surface tensors as

$$\chi_{\alpha;\beta}=\mu^{\nu}_{\alpha}(\bar{\chi}_{\nu:\beta}-b_{\nu\beta}\bar{\chi}^{3})\qquad \chi^{\alpha}{}_{;\beta}=(\overset{-1}{\mu})^{\alpha}_{\nu}(\bar{\chi}^{\nu}{}_{:\beta}-b^{\nu}_{\beta}\bar{\chi}^{3})$$

$$\chi_{\alpha;3}=\mu^{\nu}_{\alpha}\bar{\chi}_{\nu,3},\qquad \chi_{3;\alpha}=\bar{\chi}_{3,\alpha}+b^{\beta}_{\alpha}\bar{\chi}_{\beta} \tag{31}$$

$$\chi^{\alpha}_{;3}=(\overset{-1}{\mu})^{\alpha}_{\beta}\bar{\chi}^{\beta}{}_{,3},\qquad \chi^{3}_{;\alpha}=\bar{\chi}^{3}_{,\alpha}+b_{\alpha\beta}\bar{\chi}^{\beta}$$

$$\chi^{3}{}_{;3}=\chi_{3;3}=\chi_{3,3}=\bar{\chi}^{3}{}_{,3}=\bar{\chi}_{3,3}$$

and the identities of the form

$$\mu\mu^{\nu}_{\alpha}\chi^{\alpha\beta}{}_{;\beta}=(\mu\mu^{\nu}_{\alpha}\chi^{\sigma\beta})_{:\beta}-\mu\mu^{\nu}_{\alpha}(\overset{-1}{\mu})^{\beta}_{\sigma}b^{\sigma}_{\beta}\chi^{\alpha 3}-\mu b^{\nu}_{\beta}\chi^{3\beta}$$

$$\mu\chi^{3\alpha}{}_{;\alpha}=(\mu\chi^{3\alpha})_{:\alpha}+\mu\mu^{\alpha}_{\nu}b_{\alpha\beta}\chi^{\nu\beta}-\mu(\overset{-1}{\mu})^{\alpha}_{\nu}b^{\nu}_{\alpha}\chi^{33} \tag{32}$$

$$\mu_\alpha^\beta \chi^{\alpha 3}{}_{;\alpha} = (\mu_\alpha^\beta \chi^{\alpha 3})_{,3} \quad , \quad \mu \chi^\alpha{}_{;\alpha} = (\mu \chi^\alpha)_{:\alpha} + \mu_{,3} \chi^3 ,$$
$$\mu_{,3} = -\mu (\overset{-1}{\mu})^\alpha_\beta \, b^\beta_\alpha \tag{32}$$

are recorded for later use. Here and henceforth, colons are used to designate covariant derivatives with respect to the indicated coordinate by use of surface metrics and semicolons those by use of space metrics. A more elaborate account of preliminaries from the differential geometry of a surface may be found [33].

Further, the elements of volume dV, of surface dS on S, of area dA on A, and of line ds along C are of the forms

$$dV = \sqrt{g}\, dx^1 dx^2 dx^3 = \mu dA\, dx^3 = dS\, dx^3 \quad , \quad n_\alpha\, dS = \mu \nu_\alpha ds\, dx^3 \tag{33}$$

with

$$\mu = |\mu^\alpha_\beta| = (g/a)^{1/2} = 1 - 2x^3 K_m + (x^3)^2 K_g$$
$$a = |a_{\alpha\beta}| , \quad g = |g_{ij}| ; \tag{34}$$

and

$$K_m = \frac{1}{2} b^\alpha_\alpha \quad , \quad K_g = |b^\alpha_\beta| = b \tag{35}$$

where K_m and K_g are the mean and Gaussian curvatures of the reference surface A. In the foregoing relatins, μ^α_β and its inverse $(\overset{-1}{\mu})^\alpha_\beta$ are of particular importance. They play the role of shifters between space and surface tensors, and they do exist when

$$|x^3| < |R_{min}| \tag{36}$$

where R_{min} denotes the least principal radius of curvature of A; this sufficient condition is evidently satisfied by the fundamental assumption of the form

$$2H/|R_{min}| << 1 \tag{37}$$

for the laminae region.

4- A DIRECT METHOD OF SOLUTION

In this section, a direct method of solution which is essentially based on Kantorovich's method for the mechanical displacements and the electric potential is presented for the macromechanical analysis of piezoelectric strained laminae. The series expansions below has freedom to account for all the significant mechanical and electrical effects in each constituent as well as for the dynamic internations of adherent constituents. The unified method of solution provides an alternative for the vibration analysis of piezoelectric laminae under a mechanical bias (cf., [5]-[9]).

Mechanical displacements and electric potential

In mathematical terms, the fundamental assumption (37) defines the laminae region and it allows to treat the laminae region as a two-dimensional continuum. In addition to (37), suitable regularity, smoothness and absence of any kind of singularities are considered for the laminae region. All the field quantities together with their derivatives are taken to be exist and to be single-valued and continuous functions of the space coordinates and time in the closure of region $\bar{V}$, and not to vary widely across the thickness of laminae constituents. In accordance with these assumptions, the fields of incremental mechanical displacements and electric potential which are chosen as a starting point of derivation are represented, applying Weierstrass's theorem, by the series expansions in thickness coordinate as

$$\{\bar{u}_i(x^j,t),\phi(x^i,t)\}^{(m)}=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\{f_i^{pqr}(x^\alpha,t)\,f_r(x^3),\; g_{pqr}(x^\alpha,t)\,g_r(x^3)\}^{(m)} \tag{38a}$$

for the m-th constituent. Here, $\bar{u}_i$ stands for the shifted components of displacements defined by (29), and f_i^{pqr} and g_{pqr} for coordinate or approximating functions, the system of which is assumed to be complete, and they are expressed by

$$\{f_i^{pqr},g_{pqr}\}=\{\alpha_{pqr}^{(i)}(t)\,u_i^{pq}(x^\alpha),\;\gamma_{pqr}(t)\,\phi_{pq}(x^\alpha)\} \tag{38b}$$

The functions u_{pq} and ϕ_{pq} are chosen appropriately to satisfy all or some of the prescribed displacement and electric potential boundary conditions. Also, the functions f_r and g_r are known *a priori*, whereas α_{pqr} and δ_{pqr} are

functions to be determined. The functions u_{pq} and ϕ_{pq} should be chosen as simple as possible so that operations involving them can be carried out easily. They may be chosen as products of power series and trigonometric series (or surface harmonics, Legendre polynomials and alike) multiplied by certain functions which are introduced to satisfy the boundary conditions. Further, a truncated form of the expansions above, namely,

$$\{\bar{u}_i,\phi\}^{(m)}=\sum_{p=0}^{P}\sum_{q=0}^{Q}\sum_{r=0}^{R}\{\alpha_{pqr}^{(i)}\,u_i^{pq}\;,\;\gamma_{pqr}\phi_{pq}\}^{(m)}(x^3)^r \quad (39a)$$

with

$$f_r=g_r=(x^3)^r;\ \alpha_{pqr}^{(1)}=\alpha_{pqr}\;,\ \alpha_{pqr}^{(2)}=\beta_{pqr}\;,\ \alpha_{pqr}^{(3)}=\nu_{pqr} \quad (39b)$$

is considered. In (39), $\mathcal{N}=P+Q+R$ may be called the order of approximation.

C o n t i n u i t y c o n d i t i o n s

At the interfaces of laminae constituents, the continuity conditions of tractions which result from Newton's third law of mechanics are given by

$$\tau_{(m)}^{i}+\tau_{(m+1)}^{i}=0;\quad m=1,2,...,N-1\quad \text{on}\quad S_{m,m+1}XT \quad (40)$$

where $S_{m.m+1}$ denotes the bonding surface between the m-th and (m+1)-th constituents. On the other hand, the continuity of mechanical displacements depends on the manufacturing process of laminae, and the constituents of laminae are assumed herein to be perfectly bonded, and hence the continuity conditions are expressed by

$$\bar{u}_i^{(m)}-\bar{u}_i^{(m+1)}=0\ ;\ m=1,2,...,N-1\quad \text{on}\quad S_{m,m+1}\ XT \quad (41)$$

Moreover, the continuity of surface charge and that of electric potential are stated by

$$\sigma_{(m)}+\sigma_{(m+1)}=0\ ,\ \phi_{(m)}-\phi_{(m+1)}=0\ ;\ m=1,2,...,N-1\ \text{on}\quad S_{m,m+1}\ XT \quad (42)$$

on the bonding surfaces of laminae.

Now, using equations (39) and (41) the continuity of mechanical displacements is expressed by

$$\sum_{p,q,r=0}^{P,Q,R} \{[\alpha_{pqr}^{(i)} u_i^{pq}(z+h)^r]_{(m)} - [\alpha_{pqr}^{(i)} u_i^{pq}(z-h)^r]_{(m+1)}\}=0 \tag{43}$$

This equation expresses point by point continuity of incremental displacements at the interface $S_{m,m+1}$, and it can be hardly satisfied. Nevertheless, with the help of an averaging procedure, equation (43) may be stated in a more suitable form by

$$\sum_{p,q,r=0}^{P,Q,R} V_{pqr}^{i(m)}=0; \quad i=1,2,3 \quad \text{and} \quad m=1,2,\ldots,N-1 \tag{44a}$$

with

$$V_{pqr}^{i(m)} = \int_T [(\alpha_{pqr}^{(i)} U_i^{pqr})^{(m)} - (\alpha_{pqr}^{(i)} U_i^{pqr})^{(m+1)}]dt \tag{44b}$$

where

$$U_i^{pq} = \int_A u_i^{pq}\, dA \;;\; (U_i^{\prime pqr}, U_i^{\prime\prime pqr}) = \mu U_i^{pq}(x^3)^r \text{ at } x^3(z+h,z-h) \tag{44c}$$

The continuity of electric potential is written as follows

$$\sum_{p,q,r=0}^{P,Q,R} V_{pqr}^{(m)} = 0 \quad ; \quad m=1,2,\ldots,N-1 \tag{45a}$$

with

$$V_{pqr}^{(m)} = \int_T |(\Phi'_{pqr}\gamma_{pqr})^{(m)} - (\Phi''_{pqr}\gamma_{pqr})^{(m+1)}|dt \tag{45b}$$

where

$$\Phi_{pq} = \int_A \phi_{pq} dA \;;\; (\Phi'_{pqr}, \Phi''_{pqr}) = \mu\phi_{pq}(x^3)^r \quad \text{at } x^3=(z+h,z-h) \tag{45c}$$

is introduced as in (45) and equations (39) and (42) are used.

The continuity conditions (40,42) are explicitly given in the next section.

Initial conditions

In view of equations (13), the initial conditions are expressed by

$$\{(V_i^*, W_i^*) - \sum_{p,q,r=0}^{P,Q,R} U_i^{pqr}[\alpha_{pqr}^{(i)}(t_o),\ \dot{\alpha}_{pqr}^{(i)}(t_o)]\}^{(m)} = 0$$

$$[\phi^* - \sum_{p,q,r=0}^{P,Q,R} \Phi_{pqr}\gamma_{pqr}(t_o)]^{(m)} = 0 \tag{46}$$

where

$$(V_i^*, W_i^*, \Phi^*) = \int_A dA \int_Z (v_i^*, \omega_i^*, \psi^*)\ \mu dx^3$$

$$(U_i^{pqr}, \Phi_{pqr}) = \int_Z (U_i^{pq}, \phi_{pq})\mu(x^3)^r dx^3 \tag{47}$$

are introduced.

Stresses and Electric Displacements

With the help of equations (3)-(5), one obtains the components of stress tensor as

$$t^{ij} = C^{ijkl}\ u_{k;l} + C^{kij}\phi_{,k} \tag{48}$$

and inserting (39) into this equation, the components are written in the form

$$t^{ij} = \sum_{p,q,r=0}^{P,Q,R} (u_{pqr}^{ij1}\alpha_{pqr} + u_{pqr}^{ij2}\beta_{pqr} + u_{pqr}^{ij3}\ \nu_{pqr} + \phi_{pqr}^{ij}\gamma_{pqr})(x^3)^r \tag{49a}$$

$$u_{pqr}^{ij1} = [(r+1)C^{ij13} - (r-1)C^{ij\sigma 3}b_\sigma^1]u_1^{pq}$$
$$+[C^{ij1\beta} - C^{ij\sigma\beta}(1-\delta_{0,r})b_\sigma^1]u_{1:\beta}^{pq}$$
$$u_{pqr}^{ij2} = [(r+1)C^{ij23} - (r-1)C^{ij\sigma 3}b_\sigma^2]u_2^{pq} \tag{49b}$$
$$+[C^{ij2\beta} - C^{ij\sigma\beta}(1-\delta_{0,r})b_\sigma^2]u_{2:\beta}^{pq}$$
$$u_{pqr}^{ij3} = \{C^{ij\alpha\beta}[-b_{\alpha\beta} + (1-\delta_{0,r})c_{\alpha\beta}]$$
$$+C^{ij33}(r+1)\}\ u_3^{pq} + C^{ij\alpha 3}(r+1)u_{3,\alpha}^{pq}$$

and

$$\phi^{ij}_{pqr}=C^{\alpha ij}\phi_{pq,\alpha}+C^{3ij}(r+1)\phi_{pq} \tag{49c}$$

Similarly, the components of electric displacements are given by

$$D^{i}=C^{ijk}u_{j;k}-C^{ij}\phi_{,j} \tag{50}$$

where equations (4) and (6) are considered. Substituting (39) into (50), one obtains the components as

$$D^{i}=\sum_{p,q,r=0}^{P,Q,R}(u^{i1}_{pqr}\alpha_{pqr}+u^{i2}_{pqr}\beta_{pqr}+u^{i3}_{pqr}\nu_{pqr}+\phi^{i}_{pqr}\gamma_{pqr})(x^{3})^{r} \tag{51a}$$

where

$$\begin{aligned}
u^{i1}_{pqr}&=[(r+1)C^{i13}-(r-1)C^{i\sigma 3}b^{1}_{\sigma}]u^{pq}_{1}\\
&\quad+[C^{i1\beta}-C^{i\sigma\beta}(1-\delta_{o,r})b^{1}_{\sigma}]u^{pq}_{1:\beta}\\
u^{i2}_{pqr}&=[(r+1)C^{i23}-(r-1)C^{i\sigma 3}b^{2}_{\sigma}]u^{pq}_{2}\\
&\quad+[C^{i2\beta}-C^{i\sigma\beta}(1-\delta_{o,r})b^{2}_{\sigma}]u^{pq}_{2:\beta}\\
u^{i3}_{pqr}&=\{C^{i\alpha\beta}[-b_{\alpha\beta}+(1-\delta_{o,r})c_{\alpha\beta}]\\
&\quad+C^{i33}(r+1)\}u^{pq}_{3}+C^{i\alpha 3}(r+1)u^{pq}_{3,\alpha}
\end{aligned} \tag{51b}$$

and

$$\phi^{i}_{pqr}=C^{\alpha i}\phi_{pq,\alpha}+C^{3i}(r+1)\phi_{pq} \tag{51c}$$

are defined.

C o n t i n u t y   c o n d i t i o n s

The continuity conditions of tractions (40) are expressed by

$$\sum_{p,q,r=0}^{P,Q,R}\left[\sum_{j=1}^{3}(C^{i(j)}_{pqr}+C^{i(j)}_{o,pqr})\alpha^{j}_{pqr}+C^{i}_{pqr}\gamma_{pqr}\right]^{(m)}=0;\quad m=1,2,\ldots,N-1 \tag{52a}$$

with

$$C^{ij(m)}_{kln}=q^{ij(m)}_{kln}-p^{ij(m+1)}_{kln}, \quad C^{ij(m)}_{o,kln}=q^{ij(m)}_{o,kln}-p^{ij(m+1)}_{o,kln}$$
$$C^{i(m)}_{kln}=q^{i(m)}_{kln}-p^{i(m+1)}_{kln} \tag{52b}$$

where

$$(q^{\alpha i}_{kln},p^{\alpha i}_{kln})=\int_A \mu(u^{3\alpha i}_{kln}-x^3 b^{\alpha}_{\beta} u^{3\beta i}_{kln})(x^3)^n dA$$
$$(q^{3i}_{kln},p^{3i}_{kln})=\int_A \mu\ u^{33i}_{kln}(x^3)^n dA \quad \text{at} \quad x^3=(z+h,z-h) \tag{53a}$$
$$(q^{i}_{kln},p^{i}_{kln})=\int_A \mu\ \phi^{3i}_{kln}(x^3)^n dA$$

and

$$(q^{\alpha\sigma}_{o,kln};p^{\alpha\sigma}_{o,kln})=\int_A \mu(x^3)^{n-1}(nt^{33}_o u^{\sigma}_{kl}+x^3 t^{3\nu}_o u^{\sigma}_{kl:\nu})dA$$
$$(q^{\alpha 3}_{o,kln};p^{\alpha 3}_{o,kln})=-\int_A \mu(x^3)^{n-1}\ t^{3\nu}_o b^{\alpha}_{\nu}\ u^{pq}_3\ dA$$
$$\text{at} \quad x^3=(z+h,z-h) \tag{53b}$$
$$(q^{3\alpha}_{o,kln};p^{3\alpha}_{o,kln})=\int_A \mu(x^3)^n t^{3\nu}_o b^{\alpha}_{\nu} u^{kl}_{(\alpha)}\ dA$$
$$(q^{33}_{o,kln},p^{33}_{o,kln})=\int_A \mu(x^3)^{n-1}(n\ t^{33}_o u^{kl}_3+x^3 t^{3\alpha}_o u^{kl}_{3,\alpha})dA$$

at the interfaces $S_{m,m+1}$ between the layers (m) and (m+1). Likewise, the continuity of surface charge at $S_{m,m+1}$ is written in the form

$$\sum_{p,q,r=0}^{P,Q,R}\left(\sum_{i=1}^{3} d^{(i)}_{pqr}\alpha^{i}_{pqr}+d_{pqr}\gamma_{pqr}\right)^{(m)}=0 \tag{54a}$$

with

$$d^{i(m)}_{pqr}=f^{i(m)}_{pqr}-e^{i(m+1)}_{pqr}, \quad d^{(m)}_{pqr}=f^{(m)}_{pqr}-e^{(m+1)}_{pqr} \tag{54b}$$

where

$$(f^{i}_{kln},e^{i}_{kln})=\int_A \mu u^{3i}_{kln}\ (x^3)^n dA$$
$$\text{at} \quad x^3=(z+h,z-h) \tag{55}$$
$$(f_{kln},e_{kln})=\int_A \mu\phi^3_{kln}\ dA$$

Here, the relationships (31) and the identities (32) are used.

V a r i a t i o n a l   i n t e g r a l   of
i n c r e m e n t a l   m o t i o n

The equations of incremental motion (16) are written for a constituent in the form

$$\delta J_1 = \int_T dt \int_B \mathcal{L}^i \delta u_i dV = \int_T dt \int_A dA \int_Z [(t^{ij} + t_o^{ik} u^j{}_{;k})_{;i} - \rho a^j] \delta u_j \, \mu dx^3 = 0 \qquad (56)$$

By using the identities (30)-(32) and (54) and the relationships (33)-(35), equation (56) can be written in terms of the shifted components of incremental mechanical displacements as follows.

$$\begin{aligned}
\delta J_1 = \int_T dt \int_A dA \int_Z \Big\langle \{ & (\mu t^{\alpha\beta} \mu_\beta^\delta)_{:\alpha} - b_\alpha^\delta \mu t^{\alpha 3} + (\mu t^{3\alpha} \mu_\alpha^\delta)_{,3} \\
& + [\mu t_o^{\alpha\beta} (\bar{u}^\delta_{:\beta} - b_\beta^\delta \bar{u}_3) + \mu t_o^{\alpha 3} \bar{u}^\delta_{,3}]_{:\alpha} \\
& - b_\alpha^\delta [\mu t_o^{\alpha\beta} (\bar{u}_{3,\beta} + b_\beta^\lambda \bar{u}_\lambda) + \mu t_o^{\alpha 3} \bar{u}_{3,3}] \\
& + [\mu t_o^{3\alpha} (\bar{u}^\delta{}_{:\alpha} - b_\alpha^\delta \bar{u}_3) + \mu t_o^{33} \bar{u}^\delta{}_{,3}]_{,3} - \mu \mu_\alpha^\delta \rho \bar{\ddot{u}}^\alpha \} \delta \bar{u}_\delta \\
+ \{ (\mu t^{\alpha 3})_{:\alpha} & + b_{\alpha\beta} \mu \mu_\delta^\alpha t^{\beta\delta} + (\mu t^{33})_{,3} + [\mu t_o^{\alpha\beta} (\bar{u}_{3,\beta} + b_\beta^\delta \bar{u}_\delta) \\
& + \mu t_o^{\alpha 3} \bar{u}_{3,3}]_{:\alpha} \\
& + b_{\alpha\beta} [\mu t_o^{\beta\delta} (\bar{u}^\alpha_{:\delta} - b_\delta^\alpha \bar{u}_3) + \mu t_o^{\beta 3} \bar{u}^\alpha{}_{,3}] \\
& + [\mu t_o^{3\alpha} (\bar{u}_{3,\alpha} + b_\alpha^\beta \bar{u}_\beta) + \mu t_o^{33} \bar{u}_{3,3}]_{,3} \\
& - \mu \rho \bar{\ddot{u}}^3 \} \delta \bar{u}_3 \Big\rangle dx^3 = 0 \qquad (57)
\end{aligned}$$

By inserting the expansions (39) into this equation and then performing integrations over the thickness of constituent, one reads

$$\delta J_1 = \int_T dt \int_A \sum_{p,q,r=0}^{P,Q,R} \sum_{j=1}^{3} \{(T_r^{\sigma j} - b_\beta^\nu \delta_\nu^j T_{r+1}^{\sigma\beta})_{:\sigma}$$

$$+ \left[-b_\nu^\sigma \delta_\sigma^j T_r^{\nu 3} + \delta_{(j)}^3 (b_\nu^\sigma \delta_\sigma^j T_r^{\nu 3} + b_{\nu\beta} T_r^{\nu\beta} - c_{\nu\beta} T_{r+1}^{\nu\beta})\right]$$

$$+ \left[-r(T_{r-1}^{3j} - b_\nu^\sigma \delta_\sigma^j T_r^{3\nu}) + P_r^j - Q_r^j\right]\} u_{(j)}^{pq} \delta\alpha_{pqr}^j \, dA$$

$$+ \int_T dt \int_A \sum_{p,q,r=0}^{P,Q,R} \sum_{k,l,n=0}^{P,Q,R} \sum_{j=1}^{3} \sum_{i=1}^{3} (T_{pqrkln}^{oij} + R_{pqrkln}^{oij}) \alpha_{kln}^{(i)} \delta\alpha_{pqr}^{(j)}$$

$$- \int_T dt \int_A \sum_{p,q,r=0}^{P,Q,R} \sum_{k,l,n=0}^{P,Q,R} \sum_{j=1}^{3} \rho (A_{pqrkln}^j \ddot{\alpha}_{kln}^j - A_{pqrkln}^{j\nu} \ddot{\alpha}_\nu^{kln}) \delta\alpha_{pqr}^{(j)} dA = 0 \tag{58}$$

with the denotations of the form

$$(T_r^{ij}, T_{or}^{ij}) = \int_Z \mu (t^{ij}, t_o^{ij}) (x^3)^r dx^3$$

$$\mu_r = \int_Z \mu (x^3)^r \, dx^3 \tag{59}$$

$$A_{pqrkln}^j = \mu_{n+r} u_{kl}^j u_{(j)}^{pq} \quad , \quad A_{pqrkln}^{j\nu} = \mu_{n+r+1} \delta_\sigma^j b_\nu^\sigma u_{kl}^{(\nu)} u_{(j)}^{pq}$$

$$(P_r^i, Q_r^i) = \mu (t^{3i} - x^3 b_\nu^\sigma \delta_\sigma^i \, t^{3\nu})(x^3)^r \quad \text{at} \quad x^3 = (z+h, z-h)$$

$$R_r^i = P_r^i - Q_r^i$$

and

$$T_{pqrkln}^{o\nu\lambda} = (T_{o,n+r}^{\sigma\beta} \, u_{kl:\beta}^\nu + n T_{o,n+r-1}^{\sigma 3} u_{kl}^\nu)_{:\sigma} u_{pq}^{(\nu)}$$

$$- b_\sigma^\nu T_{o,n+r}^{\sigma\beta} \, b_\sigma^{(\lambda)} u_{kl}^\lambda u_{pq}^{(\nu)} - r(T_{o,n+r-1}^{3\sigma} u_{kl:\sigma}^\nu + n T_{o,n+r-2}^{33} u_{kl}^\nu) u_{(\nu)}^{pq}$$

$$T_{pqrkln}^{o3\nu} = -(T_{o,n+r}^{\sigma\beta} \, b_\beta^\nu \, u_{kl}^3)_{:\sigma} u_{pq}^{(\nu)} + r T_{o,n+r-1}^{3\sigma} b_\sigma^\nu u_{kl}^3 u_{(\nu)}^{pq}$$

$$+ \left[-b_\sigma^\nu (T_{o,n+r}^{\sigma\beta} u_{kl,\beta}^3 + n T_{o,n+r-1}^{\sigma 3} \, u_{kl}^3\right] u_{pq}^{(\nu)}$$

$$T_{pqrkln}^{o\nu 3} = (T_{o,n+r}^{\sigma\beta} \, b_\beta^\nu u_{kl}^{(\nu)})_{:\sigma} u_{pq}^3 - r T_{o,n+r-1}^{3\sigma} b_\sigma^{(\nu)} u_{kl}^\nu \, u_{pq}^3$$

$$+ (b_{\nu\beta} T_{o,n+r}^{\beta\sigma} \, u_{kl:\sigma}^{(\nu)} + n T_{o,n+r-1}^{\beta 3} \, b_{\nu\beta} u_{kl}^{(\nu)}) u_{pq}^3 \tag{60a}$$

$$T^{o33}_{pqrkln}=(T^{\sigma 3}_{o,n+r}\, u^3_{kl,\sigma}+nT^{\alpha 3}_{o,n+r-1}u^3_{kl}):_\gamma u^3_{pq}-c_{\beta\nu}T^{\beta\nu}_{o,n+r}\, u^3_{kl}u^3_{pq}$$

$$-r(T^{3\nu}_{o,n+r-1}u^3_{kl,\nu}+nT^{33}_{o,n+r-2}\, u^3_{kl})u^3_{pq}$$

and

$$(P^i_{or},Q^i_{or})=\mu t^{3i}_o(x^3)^r \quad \text{at} \quad x^3=(z+h,z-h)$$

$$R^i_{or}=P^i_{or}-Q^i_{or}$$

$$(P^{o\nu 3}_{pqrkln},\; Q^{o\nu 3}_{pqrkln})=(P^\sigma_{o,n+r},Q^\sigma_{o,n+r})\; b^\nu_\sigma u^{(\nu)}_{kl}\, u^{pq}_3$$

$$(P,Q)^{o33}_{pqrkln}=(P,Q)^\sigma_{o,n+r}u^3_{kl,\sigma}+n(P,Q)^3_{o,n+r-1}\; u^3_{kl}u^{pq}_3 \qquad (60b)$$

$$(P,Q)^{o3\nu}_{pqrkln}=-(P,Q)^\sigma_{o,n+r}b^\nu_\sigma\; u^3_{kl}u^{pq}_{(\nu)} \quad \text{at} \quad x^3=(z+h,z-h)$$

$$(P,Q)^{o\nu\lambda}_{pqrkln}=(P,Q)^\sigma_{o,n+r}u^\nu_{kl:\sigma}u^{pq}_{(\nu)}\,\delta^\nu_\lambda+n(P,Q)^3_{o,n+r-1}u^\nu_{kl}u^{pq}_{(\nu)}\delta^\nu_\lambda$$

$$R^{oij}_{pqrkln}=(P-Q)^{oij}_{pqrkln}$$

In equation (58), the first term contains incremental stress resultants, the second term includes acceleration resultants and the third term accounts for mechanical bias, that is, initial stress resultants. The incremental stress resultants in the form

$$T^{ij}_r=\sum_{k,l,n=0}^{P,Q,R}\mu_{n+r}\Big(\sum_{s=0}^{3}u^{ij(s)}_{kln}\;\alpha^{(s)}_{kln}+\phi^{ij}_{kln}\;\gamma_{kln}\Big) \qquad (61)$$

and the load resultants by

$$(P,Q)^i_r=\sum_{kln=0}^{P,Q,R}\Big(\sum_{s=1}^{3}(p,q)^{i(s)}_{klnr}\;\alpha^{(s)}_{kln}+(p,q)^i_{klnr}\gamma_{kln}\Big) \qquad (62a)$$

with

$$(p,q)^{i(s)}_{klnr}=\big[(\mu',\mu'')_{n+r}u^{3i(s)}_{kln}-(\mu',\mu'')_{n+r+1}b^\sigma_\nu\delta^i_\sigma u^{3\nu(s)}_{kln}\big]$$

$$(p,q)^i_{klnr}=\big[(\mu',\mu'')_{n+r}\phi^{3i}_{kln}-(\mu',\mu'')_{n+r+1}b^\sigma_\nu\delta^i_\sigma\phi^{3\nu}_{kln}\big] \qquad (62b)$$

$$(\mu', \mu'')_r = \mu(x^3)^r \quad \text{at} \quad x^3 = (z+h, z-h) \tag{62b}$$

and

$$R_r^i = \sum_{k,l,n=0}^{P,Q,R} \{ \sum_{s=1}^{3} r_{klnr}^{i(s)} \alpha_{kln}^{(s)} + r_{klnr}^{i} \gamma_{kln} \}$$

$$r_{klnr}^{i(s)} = (p-q)_{klnr}^{i(s)} \quad , \quad r_{klnr}^{i} = (p-q)_{klnr}^{i} \tag{62c}$$

are obtained through equations (49) and (59). Inserting (62) and (63) into the variational integral (58) and integrating over the midsurface of constituent, one finally arrives at the equation of the form

$$\begin{aligned} \delta J_1 = \int_T dt \sum_{p,q,r=0}^{P,Q,R} \sum_{k,l,n=0}^{P,Q,R,} \sum_{j=1}^{3} \Big[\sum_{i=1}^{3} (M_{pqrkln}^{ij} + M_{pqrkln}^{oij}) \alpha_{kln}^{(i)} \\ + (M_{pqrkln}^{j} + N_{pqrkln}^{j}) \gamma_{kln} \\ + \sum_{i=1}^{3} (N_{pqrkln}^{ij} + N_{pqrkln}^{oij}) \alpha_{kln}^{(i)} - \rho B_{pqrkln}^{j} \ddot{\alpha}_{kln}^{(j)} \Big] \delta \alpha_{pqr}^{(j)} \end{aligned} \tag{63}$$

for a layer. In this equation, the quantities of functions (x^α) averaged over the midsurface are defined by

$$\begin{aligned} M_{pqrkln}^{ij} = \int_A \Big[(\mu_{n+r} u_{kln}^{\sigma ji} - b_\beta^\nu \delta_\nu^i \mu_{n+r+1} u_{kln}^{\sigma\beta i})_{:\sigma} + (-b_\nu^\sigma \delta_\sigma^j \mu_{n+r} \\ + \delta_{(j)}^3 b_\nu^\sigma \delta_\sigma^j \mu_{n+r}) \; u_{kln}^{\nu 3i} \\ + \delta_j^3 (b_{\nu\beta} \mu_{n+r} - c_{\nu\beta} \mu_{n+r+1}) u_{kln}^{\nu\beta i} - r(\mu_{n+r+1} u_{kln}^{3ji} \\ - b_\nu^\sigma \delta_\sigma^j \mu_{n+r} u_{kln}^{3\nu i}) \Big] u_{(j)}^{pq} \, dA \end{aligned}$$

$$\begin{aligned} M_{pqrkln}^{j} = \int_A \Big[(\mu_{n+r} \phi_{kln}^{\sigma j} - b_\beta^\nu \delta_\nu^j \mu_{n+r+1} \phi_{kln}^{\sigma\beta})_{:\sigma} + (-b_\nu^\sigma \delta_\sigma^j \mu_{n+r} \\ + \delta_{(j)}^3 b_\nu^\sigma \delta_\sigma^j \mu_{n+r}) \phi_{kln}^{\nu 3} \end{aligned} \tag{64}$$

$$+\delta_j^3(b_{\nu\beta}\mu_{n+r}-c_{\nu\beta}\mu_{n+r+1})\phi_{kln}^{\nu\beta}-r(\mu_{n+r-1}\phi_{kln}^{3j}$$

$$-b_\nu^\sigma\ \delta_\sigma^j\mu_{n+r}\phi_{kln}^{3\nu})]u_{(j)}^{pq}\ dA$$

$$B_{pqrkln}^j=\int_A(A_{pqrkln}^j-A_{pqrkln}^{(j)\nu}\ \delta_\nu^j)dA \tag{64}$$

$$(K,L)_{pqrkln}^{ij}=\int_A(p,q)_{klnr}^{ij}u_{(j)}^{pq}dA\ ,$$

$$(K,L)_{pqrkln}^{j}=\int_A\ (p,q)_{klnr}^{j}u_{(j)}^{pq}\ dA$$

and

$$(K,L)_{pqrkln}^{oij}=\int_A\ (P,Q)_{pqrkln}^{oij}\ dA \tag{65}$$

$$(N^{ij},N^{i},N^{oij})_{pqrkln}=[(K-L)^{ij},(K-L)^{i},(K-L)^{oij}]_{pqrkln}$$

$$M_{pqrkln}^{oij}=\int_A\ T_{pqrkln}^{oij}\ dA$$

in which all the quantities are constant by definition.

Equation (63) represents the variational integral of incremental motion for a *constituent, and it is now evaluated* for all the constituents of piezoelectric laminae as follows.

$$\delta J=\sum_{m=1}^{N}\delta J_1^{(m)}=0 \tag{66}$$

With the help of equation (65), this equation can be expressed by

$$\delta J=\int_T dt\sum_{m=1}^{N}\ \sum_{p,q,r=o}^{P,Q,R}\ \sum_{k,l,n=0}^{P,Q,R}\ \sum_{j=1}^{3}(\chi_{pqrkln}^{i}\delta\alpha_{pqr}^{(j)})^{(m)} \tag{67a}$$

with

$$\chi_{pqrkln}^{j(m)}=\Big[\sum_{i=1}^{3}(M_{pqrkln}^{ij}+M_{pqrkln}^{oij})\alpha_{kln}^{(i)}+M_{pqrkln}^{j}\gamma_{kln}\Big]^{(m)}$$

$$\Big[\sum_{i=1}^{3}(K_{pqrkln}^{ij}+K_{pqrkln}^{oij})\alpha_{kln}^{(i)}\Big]^{(N)}$$

$$-\Big[\sum_{i=1}^{3}(L_{pqrkln}^{ij}+L_{pqrkln}^{oij})\alpha_{kln}^{(i)}\Big]^{(1)} \tag{67b}$$

$$+(K^{j}_{pqrkln}\gamma_{kln})^{(N)}-(L^{j}_{pqrkln}\gamma_{kln})^{(1)}-$$
$$-(\rho B^{j}_{pqrkln}\ddot{\alpha}^{(j)}_{kln})^{(m)} \qquad (67b)$$

In equation (67), the continuity conditions of tractions (52) are taken into account

V a r i a t i o n a l b o u n d a r y C o n d i t i o n s of t r a c t i o n s

Paralleling to the above derivation, the associated natural boundary conditions of tractions are established. The tractions are taken to be specified on the edge boundary surface S_e, while the displacements (39) are prescribed on the faces, and hence the variational surface integral (16) is written in the form.

$$\delta J^{\star}=\int_T dt\int_{\partial B}\overset{i}{\star}\delta u_i dS=\int_T dt\oint_C\sum_{m=1}^{N}\int_{Z_m}\nu_\alpha\Big[\tau^{\alpha j}_{\star}$$
$$-(t^{\alpha j}+t^{\alpha k}_{o}u^{j}_{;k})\Big]^{(m)}\delta u^{(m)}_{j}\,\mu dc dx^3 \qquad (68)$$

By using (30,(31),(39) and (62), one obtains

$$\delta J^{\star}=\int_T dt\sum_{m=1}^{N}\sum_{p,q,r=0}^{P,Q,R}\sum_{k,l,n=0}^{P,Q,R}\sum_{j=1}^{3}(\overset{\star j}{\chi}_{pqrkln}\delta\alpha^{(j)}_{pqr})^{(m)} \qquad (69a)$$

where

$$\overset{\star j(m)}{\chi}_{pqrkln}=\overset{\star j(m)}{M}_{pqr}-\Big[\sum_{i=1}^{3}(H^{ij}_{pqrkln}+H^{oij}_{pqrkln})\alpha^{(i)}_{kln}+H^{oj}_{pqrkln}\alpha^{(j)}_{kln}$$
$$+H^{j}_{pqrkln}\gamma_{kln}\Big]^{(m)} \qquad (69b)$$

with

$$H^{ij}_{pqrkln}=\oint_C\nu_\alpha(\mu_{n+r}u^{\alpha ij}_{kln}-\mu_{n+r+1}b^{\sigma}_{\beta}\;\delta^{i}_{\sigma}u^{\alpha\beta j}_{kln})u^{(j)}_{pq}\;dc$$
$$H^{j}_{pqrkln}=\oint_C\nu_\alpha(\mu_{n+r}\phi^{\alpha j}_{kln}-\mu_{n+r+1}b^{\sigma}_{\beta}\;\delta^{j}_{\sigma}\phi^{\alpha\beta}_{kln})u^{(j)}_{pq}dc \qquad (70)$$

and

$$(H^{o\alpha(\alpha)}_{pqrkln}, H^{o3\alpha}_{pqrkln}) = \oint_C \nu_\beta (T^{\beta\sigma}_{o,n+r} u^{\alpha}_{kl:\sigma},$$
$$-b^{\alpha}_{\sigma} T^{\beta\sigma}_{o,n+r+l} u^{3}_{kl}) u^{(\alpha)}_{pq} dc$$
$$(H^{o\alpha 3}_{pqrkln}, H^{o33}_{pqrkln}) = \oint_C \nu_\beta T^{\beta\sigma}_{o,n+r} (b_{\sigma\alpha} u^{(\alpha)}_{kl}, u^{3}_{kl,\sigma}) u^{3}_{pq} \, dc \quad (71)$$
$$H^{oi}_{pqrkln} = \oint_C \nu_\beta n \, T^{\beta i}_{o,n+r-1} \, u^{(i)}_{kl} u^{i}_{pq} \, dc$$
$$T^{\alpha i}_{*r} = \int_Z \mu \tau^{\alpha i}_{*} (x^3)^r dx^3 \quad , \quad M^{*i}_{pqr} = \oint_C \nu_\alpha (T^{\alpha i}_{*r} - b^{\sigma}_{\beta} \delta^{i}_{\sigma} \, T^{\alpha\beta}_{*r+1}) u^{(i)}_{pq} dc$$

are defined.

Variational integral of charge equation

As in the derivation of the variational integral of incremental motion above, the variational form of the charge equation of electrostatics (16) is expressed by

$$\delta I_1 = \int_T dt \int_B \mathcal{L} \delta\phi dv = \int_T dt \int_A dA \int_Z D^{i}_{;i} \delta\phi \mu dx^3 \quad (72)$$

for a constituent. The integration of this equation with respect to the thickness coordinate x^3 yields

$$\delta I_1 = \int_T dt \int_A dA \sum_{p,q,r=0}^{P,Q,R} (C^{\alpha}_{r:\alpha} - rC^{3}_{r-1} + e_r) \; \phi_{pq} \delta\gamma_{pqr} \quad (73a)$$

with

$$C^{i}_{r} = \int_Z \mu D^{i} (x^3)^r dx^3 \; ,$$
$$(73b)$$
$$(C_r, d_r) = \mu D^3 (x^3)^r \quad \text{at} \quad x^3 = (z+h, z-h); \quad e_r = c_r - d_r$$

where the expansion (39) and the identities (32) are used. In equation (73), the gross electric displacements and the surface charge resultants are obtained as

$$C^{i}_{r} = \sum_{k,l,n=0}^{P,Q,R} \mu_{n+r} \Big(\sum_{s=0}^{3} u^{i(s)}_{kln} \alpha^{(s)}_{kln} + \phi^{i}_{kln} \gamma_{kln} \Big)$$
$$(74)$$
$$(c_r, d_r) = \sum_{k,l,n=0}^{P,Q,R} (\mu'_{n+r}, \mu''_{n+r}) \Big(\sum_{i=1}^{3} u^{3i}_{kln} \alpha^{(i)}_{kln} + \phi^{3}_{kln} \gamma_{kln} \Big)$$

Next, substituting this equation into the variational integral (73) and considering it for all the constituents of piezoelectric laminae, one arrives at

$$\delta I = \int_T dt \sum_{m=2}^{N-1} \sum_{p,q,r=0}^{P,Q,R} \sum_{k,l,n=0}^{P,Q,R} (\chi_{pqrkln}\delta\gamma_{pqr})^{(m)} \tag{75}$$

with

$$\begin{aligned}\chi^{(m)}_{pqrkln} &= \Big(\sum_{i=1}^{3} E^{i}_{pqrkln}\alpha^{(i)}_{kln} + E_{pqrkln}\gamma_{kln}\Big)^{(m)} \\ &+ \Big(\sum_{i=1}^{3} D^{i}_{pqrkln}\alpha^{(i)}_{kln} + D_{pqrkln}\gamma_{kln}\Big)^{(N)} \\ &- \Big(\sum_{i=1}^{3} C^{i}_{pqrkln}\alpha^{(i)}_{kln} + C_{pqrkln}\gamma_{kln}\Big)^{(1)}\end{aligned} \tag{76}$$

and

$$(E^{i},E)_{pqrkln} = \int_A [\mu_{n+r}(u^{\alpha i},\phi^{\alpha})_{kln:\alpha} - r\mu_{n+r-1}(u^{3i},\phi^{3})_{kln}]\phi_{pq}dA$$

$$(D,C)^{i}_{pqrkln} = \int_A \mu_{n+r}u^{3i}_{kln}\phi_{pq}dA \quad \text{at} \quad x^3 = (z+h, z-h) \tag{77}$$

$$(D,C)_{pqrkln} = \int_A \mu_{n+r}\phi^{3}_{kln}\phi_{pq}\, dA$$

Here, the continuity of surface charge across interfaces of constituents (54) is included.

Variational electrical boundary conditions

The electric potential is applied to the faces of, and the surface charges are prescribed on the edge boundary surface of piezoelectric laminae. Thus, the electrical boundary conditions (16) are written as

$$\delta I_{\star} = \int_T dt \int_{\partial B} \mathcal{L}_{\star}\delta\phi dS = \int_T dt \oint_C \sum_{m=1}^{N} \int_{Z_m} (\sigma_{\star} - \nu_{\alpha}D^{\alpha})^{(m)}\delta\phi^{(m)}\mu dc dx^3 \tag{78}$$

As before, with the aid of equation (39), the evaluation of this variational integral yields

$$\delta I_{*}=\int_{T} dt \sum_{m=1}^{N} \sum_{p,q,r=0}^{P,Q,R} \sum_{k,l,n=0}^{P,Q,R} (\chi^{*}_{pqrkln}\delta\gamma_{pqr})^{(m)} \tag{79}$$

where

$$\chi^{*}_{pqrkln}=D^{*}_{pqr}-(\sum_{i=1}^{3} G^{(i)}_{pqrkln}+G_{pqrkln}) \tag{80}$$

with

$$\Lambda^{*}_{r}=\int_{Z} \sigma_{*}(x^{3})^{r} \mu dx^{3}, \qquad D^{*}_{pqr}=\oint_{C} \Lambda^{*}_{r}\phi_{pq} \, dc$$

$$(F^{i}_{pqrkln},F_{pqrkln})=\oint_{C}\nu_{\alpha}\mu_{n+r}(u^{\alpha i}_{kln},\phi^{\alpha}_{kln})dc \tag{81}$$

$$G^{i}_{pqrkln}=F^{i}_{pqrkln}\alpha^{(i)}_{kln}, \quad G_{pqrkln}=F_{pqrkln}\gamma_{kln}$$

are introduced. The boundary conditions of electric potential are expressed on the lower and upper faces, respectively, in the form.

$$\phi_{1}=\sum_{p,q,r=0}^{P,Q,R} (\phi''_{pqr}\gamma_{pqr})^{(1)}-\phi'=0 \quad \text{on} \quad S_{1f}XT \tag{82}$$

$$\phi_{N}=\sum_{p,q,r=0}^{P,Q,R} (\phi'_{pqr}\gamma_{pqr})^{(N)}+\phi''=0 \quad \text{on} \quad S_{uf}XT \tag{83}$$

with

$$\phi'=-\phi''=\phi_{o} \cos \omega t \tag{84}$$

which clearly implies that an alternative potential difference is applied to the perfectly conducting electrodes. In equations (82) and (83), ϕ_o is a constant and ω denotes the circular frequency. On the other hand, if the electrodes are shorted, these equations are then replaced by

$$\phi'=\phi''=0 \tag{85}$$

on the faces. The boundary conditions (82)-(85) are assumed to be satisfied by (39); however, they may be easily taking into account by use of Lagrange undetermined multipliers.

System of ordinary differential equations

At this point it is desirable to return the variational integral (16) of the form

$$\delta\chi\{\alpha^{i(m)}_{pqr},\gamma^{(m)}_{pqr}\}=\delta J+\delta I+\delta J_{*}+\delta I_{*}=0 \tag{86}$$

which has the continuity conditions (44) and (45) as its constraints. This variational integral is augmented through Lagrange undetermined multipliers (λ^m_j and λ_m where $m=1,2,\ldots,N-1$ and $j=1,2,3$) so as to relax the constraint conditions as follows.

$$\delta Y=\delta\chi+\delta\sum_{m=1}^{N}\sum_{i=1}^{3}\sum_{p,q,r=0}^{P,Q,R}(\lambda^m_{(i)}V^{i(m)}_{pqr}+\lambda_m V^{(m)}_{pqr})=0 \tag{87}$$

which readily leads, after taking variations, to

$$\begin{aligned}\delta Y=\int_T dt\sum_{m=1}^{N}\Big\{&\sum_{p,q,r=0}^{P,Q,R}\sum_{k,l,n=0}^{P,Q,R}(\chi_{pqrkln}+\chi^{*}_{pqrkln}+\Lambda_{pqr})\delta\gamma_{pqr}\\&+\sum_{j=1}^{3}(\chi^{j}_{pqrkln}+\chi^{*j}_{pqrkln}+\Lambda^{j}_{pqr})\delta\alpha^{(j)}_{pqr}\\&+\sum_{j=1}^{3}V^{j}_{pqr}\,\delta\lambda_j+V_{pqr}\,\delta\lambda\Big\}^{(m)}=0\end{aligned} \tag{88a}$$

where

$$\begin{aligned}\Lambda^{(m)}_{pqr}&=-(1-\delta^{(m)}_{1})\Phi''^{(m)}_{pqr}\lambda_{m-1}+(1-\delta^{m}_{N})\Phi'^{(m)}_{pqr}\lambda_m\\ \Lambda^{j(m)}_{pqr}&=-(1-\delta^{m}_{1})u''^{j(m)}_{pqr}\lambda^{m-1}_{(j)}+(1-\delta^{m}_{N})u'^{j(m)}_{pqr}\lambda^{m}_{(j)}\end{aligned} \tag{88b}$$

Owing to the fact that the variations of (α^{jm}_{pqr} and γ^{m}_{pqr}) and those of Lagrange multipliers are independent and arbitrary, it follows from equation (88)

$$\sum_{k,l,n=0}^{P,Q,R}\Big\{\sum_{i=1}^{3}(M^{ij}_{pqrkln}+M^{oij}_{pqrkln}-H^{ij}_{pqrkln}-H^{oij}_{pqrkln})_{(m)}\alpha^{(i)m}_{kln}(t) \tag{89a}$$

$$+\left[-H^{oj}_{pqrkln}\alpha^{(j)}_{kln}(t)+(M^{j}_{pqrkln}-H^{j}_{pqrkln})\gamma_{kln}(t)\right]_{(m)}$$

$$+\left[\sum_{i=1}^{3}(K^{ij}_{pqrkln}+K^{oij}_{pqrkln})\alpha^{(i)}_{kln}(t)\right]^{(N)}-\left[\sum_{i=1}^{3}(L^{ij}_{pqrkln}+L^{oij}_{pqrkln})\alpha^{(i)}_{kln}(t)\right]^{(l)}$$

$$+\left[K^{j}_{pqrkln}\gamma_{kln}(t)\right]^{(N)}-\left[L^{j}_{pqrkln}\gamma_{kln}(t)\right]^{l}+(\Lambda^{(i)}_{pqr}\lambda_{j})^{(m)}$$

$$=-M^{*j(m)}_{pqr}+\sum_{k,l,n=0}^{P,Q,R}\left[\rho B^{j}_{pqrkln}\ddot{\alpha}^{j}_{kln}(t)\right]^{(m)} \tag{89a}$$

and

$$\sum_{k,l,n=0}^{P,Q,R}\left[\sum_{i=1}^{3}(E^{i}_{pqrkln}-F^{i}_{pqrkln})\alpha^{(i)}_{kln}(t)+(E_{pqrkln}-F_{pqrkln})\gamma_{kln}(t)\right]^{(m)}$$

$$+\sum_{i=1}^{3}\{\left[D^{i}_{pqrkln}\alpha^{(i)}_{kln}(t)+D_{pqrkln}\gamma_{kln}(t)\right]^{(N)}$$

$$-\left[C^{i}_{pqrkln}\alpha^{(i)}_{kln}(t)+C_{pqrkln}\gamma_{kln}(t)\right]^{(l)}=-D^{*}_{pqr} \tag{89b}$$

This system of ordinary differential equations together with (46) govern all the extensional, thickness and flexure as well as coupled types of incremental motions for the piezoelectric laminae under a general state of mechanical bias. This system of governing equations which is second order with respect to time is reduced to algebraic equations for a case when the incremental motions become periodic as

$$\{\alpha^{j}_{pqr}(t),\gamma_{pqr}(t)\}^{(m)}=e^{i\omega t}\{\beta^{j}_{pqr},\nu_{pqr}\}^{(m)} \tag{90}$$

In such a case, α^{j}_{pqr} is replaced by β^{j}_{pqr}, γ_{pqr} by ν_{pqr} and the left side of equation (89a) by

$$-M^{*j(m)}_{pqr}+\omega^{2}\sum_{k,l,n=0}^{P,Q,R}\left[\rho B^{j}_{pqrkln}\beta^{(j)}_{kln}\right]^{(m)} \tag{91}$$

in equations (89). Numerical solutions are available for the system of governing equations under the boundary and initial conditions prescribed for any case of interest.

5- METHOD OF MOMENTS

The method of moments is one of the universal methods of solutions using computers, and it can be applied to almost any type of field equations in differential form. Though it is very popular in electromagnetic theory, the method of moments is first described herein in piezoelectricity (cf.,[34-37]). Thus, this section is devoted to describe the method of moments for a macromechanical analysis of waves and vibrations in the piezoelectric laminae under a general state of initial stresses, as an alternative of the direct method of solution presented in the previous chapter.

Mechanical displacements and electric potential

The mechanical displacements of a constituent of the piezoelectric strained laminae are expressed by

$$\bar{u}_k(x^j,t)=[v_k(x^\alpha)+x^3 w_k(x^\alpha)]\, e^{i\omega t} \tag{92a}$$

with

$$(v_k,w_k)=\sum_{p,q=0}^{P,Q}\left[\alpha_{pq}^{(k)} v_k^{pq}(x^\alpha),\beta_{pq}^{(k)} w_k^{pq}(x^\alpha)\right] \tag{92b}$$

which is a truncated version of equations (39) for r=1. The expansion (92) is a generalization of the Kirchnoff-Love hypothesis of shells,and it leads to a shear deformable theory of shells. In accordance with (92), the electric potential of a constituent is expressed by

$$\phi(x^j,t)=[\kappa(x^\alpha)+x^3\xi(x^\alpha)]\, e^{i\omega t} \tag{93a}$$

with

$$(\kappa,\xi)=\sum_{p,q=0}^{P,Q}\left[\gamma_{pq}\kappa_{pq}(x^\alpha),\nu_{pq}\xi_{pq}(x^\alpha)\right] \tag{93b}$$

In the above equations, α_{pq}, β_{pq}, γ_{pq} and ν_{pq} are unknown coefficients to be determined. The trial (approximating) functions $(v,w)_k^{pq}$ and $(\kappa,\xi)_{pq}$ should all possess second derivatives, and they need not to

satisfy any of boundary conditions, except that they should not be zero for all (p,q) at any point in the closure $\bar{B}$ of laminae where the exact solutions are not zero. However, if the approximating functions do satisfy some of the boundary conditions, certain desirable results are achieved as pointed out by Holland and EerNisse [34]. Owing to the time dependence of the mechanical displacements and the electric potential in equations (92) and (93), henceforth the factor (exp $j\omega t$) and the integrals over T need not be considered.

In view of equations (92) and (93), the continuity conditions (41) and (42) are expressed by

$$\varepsilon_i^{(m)} = \sum_{p,q=0}^{P,Q} \Lambda_i^{pq(m)} = 0, \quad i=1,2,3 \text{ and } m=1,2,\ldots,N-1 \qquad (94)$$

$$\varepsilon^{(m)} = \sum_{p,q=0}^{P,Q} \Delta_{pq}^{(m)} = 0, \quad m=1,2,\ldots,N-1 \qquad (95)$$

where

$$\Lambda_i^{pq(m)} = (\alpha_{pq}^{(i)} V_i^{pq} + (z+h)\beta_{pq}^{(i)} W_i^{pq})^{(m)} - (\alpha_{pq}^{(i)} V_i^{pq} + (z-h)\beta_{pq}^{i} W_i^{pq})^{(m-1)}$$

$$\Delta_{pq}^{(m)} = (\gamma_{pq} K_{pq} + (z+h)\nu_{pq} L_{pq})^{(m)} - (\gamma_{pq} K_{pq} + (z-h)\nu_{pq} L_{pq})^{(m+1)} \qquad (96)$$

with

$$[(V,W)_i^{pq}; (K,L)_{pq}] = \int_A [(v,w)_i^{pq}, (\kappa,\xi)_{pq}] \, dA \qquad (97)$$

are defined as in equations (44) and (45).

S t r a i n and e l e c t r i c f i e l d d i s t r i b u t i o n s

Substitution of (92) and (93) into the gradient equations (3) and (4) gives the distributions of strain and electric field in the form

$$(S_{ij}, E_i) = \sum_{p,q=0}^{P,Q} \left(\sum_{r=0}^{2} {}_r S_{ij}^{pq}, \sum_{r=0}^{1} {}_r E_i^{pq}\right) \qquad (98)$$

where the notation

$$ {}_0S^{pq}_{\sigma\eta}=\frac{1}{2}(\alpha^{(\sigma)}_{pq}v^{pq}_{\sigma:\eta}+\alpha^{(\eta)}_{pq}\ v^{pq}_{\eta:\sigma}-2b_{\sigma\eta}\alpha^{3}_{pq}v^{pq}_{3}) $$

$$ {}_1S^{pq}_{\sigma\eta}=\frac{1}{2}[\beta^{(\sigma)}_{pq}w^{pq}_{\sigma:\eta}+\beta^{(\eta)}_{pq}w_{\eta:\sigma}-2b_{\sigma\eta}\beta^{3}_{pq}w^{pq}_{3}-\sum_{r=1}^{2}(b^{\nu}_{\eta}v_{\nu:\sigma}+b^{\nu}_{\sigma}v_{\nu:\eta}) $$

$$ +2c_{\sigma\eta}\alpha^{3}_{pq}v^{pq}_{3}] $$

$$ {}_2S^{pq}_{\sigma\eta}=\frac{1}{2}[\ \sum_{\nu=1}^{2}(b^{\nu}_{\sigma}\beta^{(\nu)}_{pq}w_{\nu:\eta}+b^{\nu}_{\eta}\beta^{(\nu)}_{pq}w_{\nu:\sigma})+2_{\sigma\eta}\beta^{3}_{pq}w^{pq}_{3}] $$

$$ {}_0S^{pq}_{\sigma3}=\frac{1}{2}\ (\beta^{(\sigma)}_{pq}w^{pq}_{\sigma}+\alpha^{3}_{pq}v^{pq}_{3,\sigma}+\ \sum_{\nu=1}^{2}b^{\nu}_{\sigma}\alpha^{(\nu)}_{pq}v^{pq}_{\nu}) \tag{99a} $$

$$ {}_1S^{pq}_{\sigma3}=\frac{1}{2}\beta^{3}_{pq}w^{pq}_{3,\sigma}\quad ,{}_2S^{pq}_{\alpha3}=0 $$

$$ {}_0S_{33}=\beta^{3}_{pq}w^{pq}_{3}\quad ,\quad {}_0S^{pq}_{33}={}_1S^{pq}_{33}\ =0 $$

and

$$ {}_0E^{pq}_{\alpha}=-\gamma_{pq}\kappa_{pq,\alpha}\ ,\ {}_1E^{pq}_{\alpha}=-\nu_{pq}\ \xi^{pq}_{,\alpha}\ ,\ {}_0E^{pq}_{3}=\nu_{pq}\ \xi_{pq},{}_1E^{pq}_{3}=0 \tag{99b} $$

are used.

S t r e s s e s and e l e c t r i c
d i s p l a c e m e n t s

By use of the distributions (98) in equations (5) and (6), one finds the components of stress tensor as

$$ t^{ij}=\sum_{p,q=0}^{P,Q}\ \sum_{r=0}^{2}({}_rv^{ij}_{pqk}\alpha^{k}_{pq}+{}_rw^{ij}_{pqk}\beta^{k}_{pq}+{}_r\kappa^{ij}_{pq}\gamma_{pq}+{}_r\xi^{ij}_{pq}\nu_{pq})(x^3)^r \tag{100} $$

and the components of electric displacements as

$$ D^{i}=\sum_{p,q=}^{P,Q}\ \sum_{r=0}^{2}({}_rv^{i}_{pqk}\alpha^{k}_{pq}+{}_rw^{i}_{pqk}\beta^{k}_{pq}+{}_r\kappa^{i}_{pq}\gamma_{pq}+{}_r\xi^{i}_{pq}\nu_{pq})(x^3)^r \tag{101} $$

where the denotations of the form

$$ {}_0v^{ij}_{pq\sigma}=C^{ij\sigma\eta}\ v^{pq}_{(\sigma:\eta)}+C^{ij\nu3}b^{\sigma}_{\nu}v^{pq}_{(\sigma)}\ ;{}_1v^{ij}_{pq\sigma}=-C^{ij\nu\eta}b^{\sigma}_{\nu}v^{pq}_{(\sigma):\eta} $$

$$ {}_0v^{ij}_{pq3}=-C^{ij\sigma\eta}b_{\sigma\eta}v^{3}_{pq}+C^{ij\sigma3}v^{pq}_{3,\sigma},\ {}_1v^{ij}_{pq3}=C^{ij\nu\eta}C_{\nu\eta}v^{3}_{pq} \tag{102a} $$

$$ {}_0w^{ij}_{pq\sigma}=C^{ij\sigma 3}w^{pq}_{(\sigma)},\ w^{ij}_{pq}=C^{ij\sigma\eta}w^{pq}_{(\sigma):\eta},\ {}_2w^{ij}_{pq\sigma}=-C^{ij\nu\eta}b^{\sigma}_{\nu}w^{pq}_{(\sigma):\eta} $$

$$ {}_0w^{ij}_{pq3}=C^{ij33}w^{pq}_{3},\ {}_1w^{ij}_{pq3}=-C^{ij\nu\eta}b_{\nu\eta}w^{pq}_{3}+C^{ij\sigma 3}w^{pq}_{3,\sigma}, \tag{102a} $$

$$ {}_2w^{ij}_{pq3}=C^{ij\sigma\nu}c_{\sigma\eta}w^{pq}_{3} $$

$$ {}_0\kappa^{ij}_{pq}=C^{\alpha ij}\kappa^{pq}_{,\alpha},\quad {}_0\xi^{ij}_{pq}=C^{3ij}\xi_{pq};\ {}_1\xi^{ij}_{pq}=C^{\alpha ij}\xi^{pq}_{,\alpha} $$

and

$$ {}_0v^{i}_{pq\sigma}=C^{i\sigma\eta}v^{pq}_{(\sigma):\eta}+C^{i\nu 3}b^{\sigma}_{\nu}v^{pq}_{(\sigma)},\ {}_1v^{i}_{pq\sigma}=-C^{i\nu\eta}b^{\sigma}_{\nu}v^{pq}_{(\sigma):\eta} $$

$$ {}_0v^{i}_{pq3}=-C^{ij\sigma\eta}b_{\sigma\eta}v^{3}_{pq}+C^{ij\sigma 3}v^{pq}_{3,\sigma},\ {}_1v^{i}_{pq3}=C^{ij\nu\eta}c_{\nu\eta}v^{3}_{pq} \tag{102b} $$

$$ {}_0w^{i}_{pq\sigma}=C^{ij\sigma 3}w^{pq}_{(\sigma)},\ {}_1w^{i}_{pq\sigma}=C^{ij\sigma\eta}w^{pq}_{(\sigma):\eta},\ {}_2w^{i}_{pq\sigma}=-C^{i\nu\eta}b^{\sigma}_{\nu}w^{pq}_{(\sigma):\eta} $$

$$ {}_0v^{i}_{pq3}=C^{ij33}w^{pq}_{3},\ {}_1w^{ij}_{pq3}=-C^{ij\nu\eta}b_{\nu\eta}w^{pq}_{3}+C^{ij\sigma 3}w^{pq}_{3,\sigma}, $$

$$ {}_2w^{i}_{pq3}=C^{i\sigma\eta}c_{\sigma\eta}w^{pq}_{3} $$

$$ {}_0\kappa^{i}_{pq}=C^{\alpha i}\kappa^{pq}_{,\alpha},\ {}_0\xi^{i}_{pq}=C^{3i}\xi_{pq},\ {}_1\xi^{i}_{pq}=C^{\alpha i}\xi^{pq}_{,\alpha} $$

with

$$ {}_2v^{ij}_{pqk}={}_2v^{i}_{pqk}=0,\quad {}_\alpha\kappa^{ij}_{pq}={}_\alpha\kappa^{i}_{pq}={}_2\xi^{ij}_{pq}={}_2\xi^{i}_{pq}=0 \tag{103} $$

are introduced.

M a c r o s c o p i c e q u a t i o n s o f i n c r e m e n t a l m o t i o n

Just as was done in the derivation of equation (58), the variational integral of incremental motion (16) by

$$ \delta J=\sum_{m=1}^{N}\left\{\int_A dA\int_Z \mathcal{L}^{i}\delta(v_i+x^3w_i)\mu dx^3\right\}^{(m)} \tag{104} $$

is expressed for the piezoelectric strained laminae. After inserting the expansion (92) into this integral and integrating over the laminae thickness, one arrives at the variational integral of the form

$$\delta J = \int_A dA \sum_{m=1}^{N} \{ [(V^k + V_o^k + l^k + l_o^k) - \rho\omega^2 a^k]\,\delta v_k + [(W^k + W_o^k + m^k + m_o^k) - \rho\omega^2 b^k]\,\delta w_k \}^{(m)} \tag{105}$$

Here, the quantities of the form

$$\begin{aligned} &V^\sigma = (N^{\alpha\sigma} - b^\sigma_\beta M^{\alpha\beta})_{:\alpha} - b^\sigma_\alpha Q^\alpha \quad , \quad V^3 = Q^\alpha_{:\alpha} + b_{\alpha\beta}N^{\alpha\beta} - c_{\alpha\beta}M^{\alpha\beta} \\ &W^\alpha = (M^{\alpha\sigma} - b^\sigma_\beta K^{\alpha\beta})_{:\alpha} - Q^\sigma \quad , \quad W^3 = R^\alpha_{:\alpha} - N + b_{\alpha\beta}M^{\alpha\beta} - c_{\alpha\beta}K^{\alpha\beta} \end{aligned} \tag{106}$$

and

$$\begin{aligned} V_o^\sigma = {} & [N_o^{\alpha\beta}(v^\sigma{}_{:\beta} - b^\sigma_\beta v_3)]_{:\alpha} + [M_o^{\alpha\beta}(w^\sigma{}_{:\beta} - b^\sigma_\beta w_3)]_{:\alpha} \\ & + (Q_o^\alpha w^\sigma)_{:\alpha} - Q_o^\alpha\, b^\sigma_\alpha w_3 \\ & - N_o^{\alpha\beta}(b^\sigma_\alpha v_{3,\beta} + b^\sigma_\alpha b^\eta_\beta v_\eta) - M_o^{\alpha\beta}(b^\sigma_\alpha w_{3,\beta} + b^\sigma_\alpha b^\eta_\beta w_\eta) \\ V_o^3 = {} & [N_o^{\alpha\beta}(v_{3,\beta} + b^\sigma_\beta v_\sigma)]_{:\alpha} - [M_o^{\alpha\beta}(w_{3,\beta} + b^\sigma_\beta w_\sigma)]_{:\alpha} + \\ & + (Q_o^\alpha\, w_3)_{:\alpha} + Q_o^\beta b_{\alpha\beta} w^\alpha \\ & + N_o^{\beta\sigma}(b_{\alpha\beta}v^\alpha{}_{:\sigma} - c_{\beta\sigma}v_3) + M_o^{\beta\sigma}(b_{\alpha\beta}w^\alpha{}_{:\sigma} - c_{\beta\sigma}w_3) \\ W_o^\sigma = {} & [M_o^{\alpha\beta}(v^\sigma{}_{:\beta} - b^\sigma_\beta v_3)]_{:\alpha} - N_o w^\sigma + [K_o^{\alpha\beta}(w^\sigma{}_{:\beta} - b^\sigma_\beta w_3)]_{:\alpha} + R^\alpha_{o:\alpha} w^\sigma \\ & - Q_o^\alpha(v^\sigma{}_{:\alpha} - b^\sigma_\alpha v_3) - M_o^{\alpha\beta}(b^\sigma_\alpha v_{3,\beta} + b^\sigma_\alpha b^\eta_\beta v_\eta) - K_o^{\alpha\beta}(b^\sigma_\alpha w_{3,\beta} + b^\sigma_\alpha b^\eta_\beta w_\eta) \\ W_o^3 = {} & [M_o^{\alpha\beta}(v_{3,\beta} + b^\sigma_\beta v_\sigma)]_{:\alpha} + [K_o^{\alpha\beta}(w_{3,\beta} + b^\sigma_\beta w_\sigma)]_{:\alpha} - Q_o^\alpha(v_{3,\alpha} + b^\beta_\alpha v_\beta) \\ & + (R^\alpha_{o:\alpha} - N_o)w_3 + M_o^{\beta\sigma}(b_{\alpha\beta}v^\alpha{}_{:\sigma} - c_{\beta\sigma}v_3) + K_o^{\beta\sigma}(b_{\alpha\beta}w^\alpha{}_{:\sigma} - c_{\beta\sigma}w_3) \end{aligned} \tag{107}$$

in terms of the stress resultants, the load resultants by

$$l^i = p^i - q^i, \quad l_o^i = p_o^i - q_o^i, \quad (m^i, m_o^i) = (p^i, p_o^i)(z+h) - (q^i, q_o^i)(z-h) \tag{108a}$$

with

$$(p^i, q^i) = \mu(t^{3i} - x^3 b_\nu^{\alpha} \delta_\alpha^i t^{3\nu})(n_3, -n_3) \quad \text{at} \quad x^3 = (z+h, z-h)$$

$$(p_o^\alpha, q_o^\alpha) = \mu\{t_o^{3\beta}[(v^\alpha_{:\beta} - b_\beta^\alpha v_3) + x^3(w^\alpha_{:\beta} - b_\beta^\alpha w_3)] + t_o^{33} w^\alpha\}$$

$$\text{at} \quad x^3 = (z+h, z-h) \tag{108b}$$

$$(p_o^3, q_o^3) = \mu\{t_o^{3\alpha}[v_{3,\alpha} + b_\alpha^\beta v_\beta + x^3(w_{3,\alpha} + b_\alpha^\beta w_\beta)] + t_o^{33} w_3\}$$

$$\text{at} \quad x^3 = (z+h, z-h)$$

the acceleration resultants by

$$a^k = \mu_o v^k + \mu_1 w^k \quad , \qquad b^k = \mu_1 v^k + \mu_2 w^k \tag{109}$$

and the stress resultants by

$$\{[N, M, K]^{\alpha\beta}, (Q, R)^\alpha, N\} = \int_Z \mu\{[1, x^3, (x^3)^2] t^{\alpha\beta}, (1, x^3) t^{\alpha 3},$$

$$t^{33}\} dx^3 \tag{110}$$

$$\{[N_o, M_o, K_o]^{\alpha\beta}, (Q_o, R_o)^\alpha, N_o\} = \int_Z \mu\{[1, x^3, (x^3)^2] t_o^{\alpha\beta}, (1, x^3) t_o^{\alpha 3},$$

$$t_o^{33}\} dx^3 \tag{111}$$

are defined. In view of equations (92), (93), (100) and (102)-(103) and after lengthy computations, one finally expresses the variational integral (105) in the form

$$\delta J = \int_A dA \sum_{m=1}^{N} \sum_{s,t=0}^{P,Q} \sum_{i=1}^{3} \sum_{p,q=0}^{P,Q} \{[(\underset{\sim}{V} + \underset{\sim}{L})_{pqk}^{(i)st} (\underset{\sim}{X})_{pq}^k$$

$$+ (\underset{\sim}{V}_o + \underset{\sim}{L}_o - \rho\omega^2 \underset{\sim}{A})_{pqk}^{(i)st} (\underset{\sim}{X}_u)_{pq}^k] \delta \alpha_i^{st} \tag{112}$$

$$+ [(\underset{\sim}{W} + \underset{\sim}{K})_{pqk}^{(i)st} (\underset{\sim}{X})_{pq}^k + (\underset{\sim}{W}_o + \underset{\sim}{K}_o - \rho\omega^2 \underset{\sim}{B})_{pqk}^{(i)st} (\underset{\sim}{X}_u)_{pq}^k] \delta \beta_i^{st}\}^{(m)}$$

Here, the column matrix of coefficients to be determined is defined by

$$(\underset{\sim}{X})^k_{pq}=(\underset{\sim}{X}^k_u,\underset{\sim}{X}_\phi)_{pq};\quad (\underset{\sim}{X}_u)^k_{pq}=(\alpha^k_{pq},\beta^k_{pq})^T,(\underset{\sim}{X}_\phi)_{pq}=(\gamma_{pq},\nu_{pq})^T \tag{113}$$

the matrix of incremental stress resultants by

$$(\underset{\sim}{V})^{(i)st}_{pqk}=(a^{(i)}_{pqk}\ b^{(i)}_{pqk}\ a^{(i)}_{pq}b^{(i)}_{pq})v^{st}_i,$$

$$(\underset{\sim}{W})^{(i)st}_{pqk}=(c^{(i)}_{pqk}\ d^{(i)}_{pqk}c^{(i)}_{pq}d^{(i)}_{pq})w^{st}_i \tag{114a}$$

with the coefficients of the form

$$a^\sigma_{pqk}=\sum_{r=0}^{2}\left[({}_rv^{\alpha\sigma}_{pqk}\mu_r-{}_rv^{\beta}_{pqk}\ b^\sigma_\beta\mu_{r+1})_{:\alpha}-{}_rv^{\alpha 3}_{pqk}\ b^\sigma_\alpha\mu_r\right]$$

$$a^3_{pqk}=\sum_{r=0}^{2}\left[({}_rv^{\alpha 3}_{pqk}\mu_r)_{:\alpha}+{}_rv^{\alpha\beta}_{pqk}b_{\alpha\beta}\mu_r-{}_rv^{\alpha\beta}_{pqk}\ c_{\alpha\beta}\mu_{r+1}\right]$$

$$a^\sigma_{pq}=\sum_{r=0}^{2}\left[({}_r\kappa^{\alpha\sigma}_{pq}\mu_r-{}_r\kappa^{\alpha\beta}_{pq}\ b^\sigma_\beta\mu_{r+1})_{:\alpha}-{}_r\kappa^{\alpha 3}_{pq}b^\sigma_\alpha\mu_r\right]$$

$$a^3_{pq}=\sum_{r=0}^{2}\left[({}_r\kappa^{\alpha 3}_{pq}\mu_r)_{:\alpha}+{}_r\kappa^{\alpha\beta}_{pq}b_{\alpha\beta}\mu_r-{}_r\kappa^{\alpha\beta}_{pq}c_{\alpha\beta}\ \mu_{r+1}\right]$$

$$b^\sigma_{pqk}=\sum_{r=0}^{2}\left[({}_rw^{\alpha\sigma}_{pqk}\mu_r-{}_rw^{\alpha\beta}_{pqk}\ b^\sigma_\beta\mu_{r+1})_{:\alpha}-{}_rw^{\alpha 3}_{pqk}b^\sigma_\alpha\mu_r\right] \tag{114b}$$

$$b^3_{pqk}=\sum_{r=0}^{2}\left[({}_rw^{\alpha 3}_{pqk}\mu_r)_{:\alpha}+{}_rw^{\alpha\beta}_{pqk}\ b_{\alpha\beta}\mu_r-{}_rw^{\alpha\beta}_{pqk}\ c_{\alpha\beta}\mu_{r+1}\right]$$

$$b^\sigma_{pq}=\sum_{r=0}^{2}\left[({}_r\xi^{\alpha\sigma}_{pq}\mu_r-{}_r\xi^{\alpha\beta}b^\alpha_\beta\mu_{r+1})_{:\alpha}-{}_r\xi^{\alpha 3}_{pq}\ b^\sigma_\alpha\mu_r\right]$$

$$b^3_{pq}=\sum_{r=0}^{2}\left[({}_r\xi^{\alpha 3}_{pq}\mu_r)_{:\alpha}+{}_r\xi^{\alpha\beta}_{pq}b_{\alpha\beta}\mu_r-{}_r\xi^{\alpha\beta}_{pq}c_{\alpha\beta}\mu_{r+1}\right]$$

and

$$c^\sigma_{pqk}=\sum_{r=0}^{2}\left[({}_rv^{\alpha\sigma}_{pqk}\mu_{r+1}-{}_rv^{\alpha\beta}_{pqk}b^\sigma_\beta\mu_{r+2})_{:\alpha}-{}_rv^{\sigma 3}_{pqk}\mu_r\right]$$

$$c^3_{pqk} = \sum_{r=0}^{2} \Big[({}_r v^{\alpha 3}_{pqk} \mu_{r+1})_{:\alpha} - {}_r v^{33}_{pqk} \mu_r + {}_r v^{\alpha\beta}_{pqk} b_{\alpha\beta} \mu_{r+1} - {}_r v^{\alpha\beta}_{pqk} c_{\alpha\beta} \mu_{r+2} \Big]$$

$$c^{\sigma}_{pq} = \sum_{r=0}^{2} \Big[({}_r \kappa^{\alpha\sigma}_{pq} \mu_{r+1} - {}_r \kappa^{\alpha\beta}_{pq} b^{\sigma}_{\beta} \mu_{r+2})_{:\alpha} - {}_r \kappa^{\sigma 3}_{pq} \mu_r \Big]$$

$$c^3_{pq} = \sum_{r=0}^{2} \Big[({}_r \kappa^{\alpha 3}_{pq} \mu_{r+1})_{:\alpha} - {}_r \kappa^{33}_{pq} \mu_r + {}_r \kappa^{\alpha\beta}_{pq} b_{\alpha\beta} \mu_{r+1} - {}_r \kappa^{\alpha\beta}_{pq} c_{\alpha\beta} \mu_{r+2} \Big]$$

$$d^{\sigma}_{pqk} = \sum_{r=0}^{2} \Big[({}_r w^{\alpha\sigma}_{pqk} \mu_{r+1} - {}_r w^{\alpha\beta}_{pqk} b^{\sigma}_{\beta} \mu_{r+2})_{:\alpha} - {}_r w^{\alpha 3}_{pqk} \mu_r \Big] \qquad (114c)$$

$$d^3_{pqk} = \sum_{r=0}^{2} \Big[({}_r w^{\alpha 3}_{pqk} \mu_{r+1})_{:\alpha} - {}_r w^{33}_{pqk} \mu_r + {}_r w^{\alpha\beta}_{pqk} b_{\alpha\beta} \mu_{r+1} - {}_r w^{\alpha\beta}_{pqk} c_{\alpha\beta} \mu_{r+2} \Big]$$

$$d^{\sigma}_{pq} = \sum_{r=0}^{2} \Big[({}_r \xi^{\alpha\sigma}_{pq} \mu_{r+1} - {}_r \xi^{\alpha\beta}_{pq} b^{\sigma}_{\beta} \mu_{r+2})_{:\alpha} - {}_r \xi^{\sigma 3}_{pq} \mu_r \Big]$$

$$d^3_{pq} = \sum_{r=0}^{2} \Big[({}_r \xi^{\alpha 3}_{pqk} \mu_{r+1})_{:\alpha} - {}_r \xi^{33}_{pqk} \mu_r + {}_r \xi^{\alpha\beta}_{pqk} b_{\alpha\beta} \mu_{r+1} - {}_r \xi^{\alpha\beta}_{pq} c_{\alpha\beta} \mu_{r+2} \Big]$$

the matrix of initial stress resultants by

$$(\underset{\sim}{V}_o)^{(i)st}_{pqk} = (a^{(i)}_{opqk} \;\; b^{(i)}_{opqk}) v^{st}_i , \quad (\underset{\sim}{W}_o)^{(i)st}_{pqk} = (c^{(i)}_{opqk} \;\; d^{(i)}_{opqk}) w^{st}_i \qquad (115a)$$

with the coefficients of the form

$$a^{(\sigma)}_{opq\sigma} = (N^{\alpha\beta}_o v^{\sigma}_{pq:\beta})_{:\alpha} - N^{\alpha\beta}_o b^{\sigma}_{\alpha} b^{(\sigma)}_{\beta} v^{pq}_{\sigma} ,$$

$$a^{(\sigma)}_{opq\eta} = -N^{\alpha\beta}_o b^{\sigma}_{\alpha} b^{\eta}_{\beta} v^{pq}_{(\eta)} (1 - \delta^{\sigma}_{\eta})$$

$$a^{(\sigma)}_{opq3} = -(N^{\alpha\beta}_o b^{\sigma}_{\beta} v^{pq}_{3})_{:\alpha} - N^{\alpha\beta}_o b^{\sigma}_{\alpha} v^{pq}_{3,\beta}$$

$$a^{(3)}_{opq\sigma} = (N^{\alpha\beta}_o \; b^{\sigma}_{\beta} v^{pq}_{(\sigma)})_{:\alpha} + N^{\beta\alpha}_o b_{\sigma\beta} v^{(\sigma)}_{pq:\alpha} \qquad (115b)$$

$$a^{(3)}_{opq3}=(N^{\alpha\beta}_{o}v^{pq}_{3,\beta})_{:\alpha}-N^{\beta\sigma}_{o}c_{\beta\sigma}v^{pq}_{3}$$

$$b^{(\sigma)}_{opq\sigma}=(M^{\alpha\beta}_{o}w^{(\sigma)}_{pq:\beta}+Q^{\alpha}_{o}\ w_{pq})_{:\alpha}-M^{\alpha\beta}_{o}b^{\sigma}_{\alpha}b^{(\sigma)}_{\beta}w^{pq}_{\sigma},$$

$$b^{(\sigma)}_{opq\eta}=-M^{\alpha\beta}_{o}b^{\sigma}_{\alpha}b^{\eta}_{\beta}w_{(\eta)}\ (1-\delta^{\sigma}_{\eta}) \qquad (115b)$$

$$b^{(\sigma)}_{opq3}=-(M^{\alpha\beta}_{o}b^{\sigma}_{\beta}w^{pq}_{3})_{:\alpha}-Q^{\alpha}_{o}b^{\sigma}_{\alpha}w^{pq}_{3}-M^{\alpha\beta}_{o}b^{\sigma}_{\alpha}w^{pq}_{3,\beta}$$

$$b^{(3)}_{opq\sigma}=(M^{\alpha\beta}_{o}b^{(\sigma)}_{\beta}w^{pq}_{\sigma})_{:\alpha}+Q^{\beta}_{o}b_{\sigma\beta}w^{(\sigma)}_{pq}\ +M^{\beta\alpha}_{o}b_{\sigma\beta}w^{(\sigma)}_{pq\ :\alpha}$$

$$b^{(3)}_{opq3}=(M^{\alpha\beta}_{o}w^{pq}_{3,\beta}+Q^{\alpha}_{o}w^{pq}_{3}\ )_{:\alpha}-M^{\beta\alpha}_{o}c_{\beta\alpha}w^{pq}_{3}$$

and

$$c^{(\sigma)}_{opq\sigma}=(M^{\alpha\beta}_{o}v^{\sigma}_{pq:\beta})_{:\alpha}-Q^{\alpha}_{o}v^{\sigma}_{pq:\alpha}-M^{\alpha\beta}_{o}b^{\sigma}_{\alpha}b^{(\sigma)}_{\beta}v^{pq}_{\sigma},$$

$$c^{(\sigma)}_{opq\eta}=-M^{\alpha\beta}_{o}b^{(\sigma)}_{\alpha}b^{\eta}_{\beta}v^{pq}_{(\eta)}\ (1-\delta^{\sigma}_{\eta})$$

$$c^{(\sigma)}_{opq3}=-(M^{\alpha\beta}_{o}b^{\sigma}_{\beta}v^{pq}_{3})_{:\alpha}+b^{\sigma}_{\alpha}Q^{\alpha}_{o}v^{pq}_{3}-M^{\alpha\beta}_{o}b^{\sigma}_{\alpha}v^{pq}_{3,\beta}$$

$$c^{(3)}_{opq\sigma}=(M^{\alpha\beta}_{o}b^{\sigma}_{\beta}v_{(\sigma)})_{:\alpha}+M^{\alpha\eta}_{o}b_{\sigma\beta}v^{(\sigma)}_{:\eta}-Q^{\alpha}_{o}b^{\sigma}_{\alpha}v_{(\sigma)}$$

$$c^{(3)}_{opq3}=(M^{\alpha\beta}_{o}v^{pq}_{3,\beta})_{:\alpha}-Q^{\alpha}_{o}v^{pq}_{3,\alpha}-M^{\beta\sigma}_{o}c_{\beta\sigma}v^{pq}_{3} \qquad (115c)$$

$$d^{(\sigma)}_{opq\sigma}=-N_{o}w^{(\sigma)}_{pq}\ +(K^{\alpha\beta}_{o}w^{\sigma}_{pq:\beta})_{:\alpha}+R^{\alpha}_{o:\alpha}w^{\sigma}-K^{\alpha\beta}_{o}b^{\sigma}_{\alpha}b^{(\sigma)}_{\beta}w_{\sigma},$$

$$d^{(\sigma)}_{opq\eta}=-K^{\alpha\beta}_{o}b^{(\sigma)}_{\alpha}b^{\eta}_{\beta}w_{(\eta)}\ (1-\delta^{\sigma}_{\eta})$$

$$d^{(\sigma)}_{opq3}=-(K^{\alpha\beta}_{o}b^{\sigma}_{\beta}w^{pq}_{3})_{:\alpha}-K^{\alpha\beta}_{o}b^{\sigma}_{\alpha}w^{pq}_{3,\beta}$$

$$d^{(3)}_{opq\sigma}=(K_o^{\alpha\beta}b^{\sigma}_{\beta}w_{(\sigma)})_{:\alpha}+K_o^{\beta\alpha}b_{\sigma\beta}w^{(\alpha)}_{pq:\alpha}$$

$$d^{(3)}_{opq3}=(K_o^{\alpha\beta}w^{pq}_{3,\beta})_{:\alpha}-c_{\beta\sigma}K_o^{\beta\sigma}w^{pq}_{3}+R^{\alpha}_{o:\alpha}w^{pq}_{3}-N_o w^{pq}_{3} \quad (115c)$$

the matrix of load resultants by

$$(\underset{\sim}{L})^{(i)st}_{pqk}=v^{st}_{i}\left[(P\ Q)^{(i)}_{pqk}(P\ Q)^{(i)}_{pq}\right],$$

$$(\underset{\sim}{K})^{(i)st}_{pqk}=w^{st}_{i}\left[(R\ S)^{(i)}_{pqk}(R\ S)^{(i)}_{pq}\right] \quad (116)$$

with the coefficients of the form

$$(P',P'';R',R'')^{i}_{pqk}=\sum_{r=0}^{2}\left[{}_{r}v^{3i}_{pqk}(\underset{\sim}{\mu}_{r}{}^{','})-{}_{r}v^{3\eta}_{pqk}b^{\sigma}_{\eta}\delta^{i}_{\sigma}(\underset{\sim}{\mu}_{r+1}{}^{','})\right]$$

$$(P',P'';R',R'')^{i}_{pq}=\sum_{r=0}^{2}\left[{}_{r}\kappa^{3i}_{pq}(\underset{\sim}{\mu}_{r}{}^{','})-{}_{r}\kappa^{3\eta}_{pq}b^{\sigma}_{\eta}\delta^{i}_{\sigma}(\underset{\sim}{\mu}_{r+1}{}^{','})\right] \quad (117)$$

$$(Q',Q'';S',S'')^{i}_{pqk}=\sum_{r=0}^{2}\left[{}_{r}w^{3i}_{pqk}(\underset{\sim}{\mu}_{r}{}^{','})-{}_{r}w^{3\eta}_{pqk}b^{\sigma}_{\eta}\delta^{i}_{\sigma}(\underset{\sim}{\mu}_{r+1}{}^{','})\right]$$

$$(Q',Q'';S',S'')^{i}_{pq}=\sum_{r=0}^{2}\left[{}_{r}\xi^{3i}_{pq}(\underset{\sim}{\mu}_{r}{}^{','})-{}_{r}\xi^{3\eta}_{pq}b^{\sigma}_{\eta}\delta^{i}_{\sigma}(\underset{\sim}{\mu}_{r+1}{}^{','})\right]$$

with

$$P^{(i)}_{pqk}=P'^{(i)}_{pqk}-P''^{(i)}_{pqk},\ \ldots,\ S^{(i)}_{pq}=S'^{(i)}_{pq}-S''^{(i)}_{pq} \quad (118)$$

$$(\underset{\sim}{\mu}_{r}{}^{','})=(\mu'_{r},\mu''_{r};\mu'_{r+1},\mu''_{r+1})$$

and

$$(\underset{\sim}{L}_{o})^{(i)st}_{pqk}=v^{st}_{i}(P_oQ_o)^{(i)}_{pqk},\quad (\underset{\sim}{K}_{o})^{(i)st}_{pqk}=w^{st}_{i}(R_oS_o)^{(i)}_{pqk} \quad (119)$$

with the coefficients of the form

$$\left[(P'_o,P''_o,R'_o,R''_o)^{\sigma}_{pq(\sigma)};(P'_o,\ldots,R''_o)^{\sigma}_{pq3}\right]=\left[(\underset{\sim}{r}^{\alpha}_{01})v^{\sigma}_{pq:\alpha};\right. \quad (120)$$

$$\left.-(\underset{\sim}{r}^{\alpha}_{01})b^{\sigma}_{\alpha}v^{3}_{pq}\right]$$

$$[(P_o',\ldots,R_o'')^3_{pq\sigma};(P_o',\ldots,R_o'')^3_{pq3}]=[(r^\alpha_{01})b^{(\sigma)}_\alpha v^\sigma_{pq};$$

$$(r^\alpha_{01})v^{pq}_{3,\alpha}] \tag{120}$$

$$[(Q_o',\ldots,S_o'')^\sigma_{pq(\sigma)};(Q_o',\ldots,S_o'')^\sigma_{pq3}]=(r^\alpha_{02})w^\sigma_{pq:\alpha}\pm(r^3_{01})w^\sigma_{pq};$$

$$-(r^\alpha_{02})b^\sigma_\alpha w^3_{pq}$$

$$[(Q_o',\ldots,S_o'')^\sigma_{pq3};(Q_o',\ldots,S_o'')^3_{pq3}]=[(r^\alpha_{01})b^\sigma_\alpha w^3_{pq};(r^\alpha_{02})b^\sigma_\alpha w^{(\sigma)}_{pq}]$$

where

$$(P_o=P_o'-P_o'')^i_{pqk},\ldots,(S_o=S_o'-S_o'')^i_{pqk} \tag{121}$$

$$(r_o^{'(i)},r_o^{''(i)})=t_o^{3i} \text{ at } x^3=(z+h,z-h)$$

$$(r^i_{0\lambda})=(\mu_\lambda' \quad \mu_\lambda'' \quad \mu_{\lambda+1}' \quad \mu_{\lambda+1}'')r^i_o$$

and the matrix of acceleration coefficients as

$$(A)^{(i)st}_{pqk}=(\mu_o v^k_{pq} \quad \mu_1 w^k_{pq})v^{st}_i,\quad (B)^{(i)st}_{pqk}=(\mu_1 v^k_{pq} \quad \mu_2 w^k_{pq})w^{st}_i \tag{122}$$

in the notation of (59).

In view of the equations above, the continuity of tractions is expressed by

$$\varepsilon^{1(m)}_{i(st)}=\sum_{p,q=0}^{P,Q}\{[L'^{(i)st}_{pqk}(X)^k_{pq}+L'^{(i)st}_{opqk}(X_u)^k_{pq}]^{(m)}$$

$$-[L''^{(i)st}_{pqk}(X)^k_{pq}+L''^{(i)st}_{opqk}(X_u)^k_{pq}]^{(m+1)}\}=0 \tag{123}$$

$$\varepsilon^{2(m)}_{i(st)}=\sum_{p,q=0}^{P,Q}\{[K'^{(i)st}_{pqk}(X)^k_{pq}+K'^{(i)st}_{opqk}(X_u)^k_{pq}]^{(m)}$$

$$-[K''^{(i)st}_{pqk}(X)^k_{pq}+L''^{(i)st}_{opqk}(X_u)^k_{pq}]^{(m+1)}\}=0$$

$$i=1,2,3\ ;\ m=1,2,\ldots,N-1$$

in which a prime (or a double prime) is used to denote the value of quantities at $x^3=z+h$ (or $x^3=z-h$), as is used in all the foregoing equations.

Mechanical boundary conditions

The boundary conditions associated with the macroscopic equations of incremental motion (112), are expressed in variational form by

$$\delta J_* = \sum_{m=1}^{N} \oint_C \left\{\int_Z \nu_\alpha \left[\tau_*^{\alpha j} - (t^{\alpha j} + t^{\alpha k} u_{j;k})\right](\delta v_k + x^3 \delta w_k)\,\mu\, dc\, dx^3\right\}^{(m)} \tag{124}$$

which clearly implies that the tractions are prescribed on the edge boundary surface and

$$\delta u_i^{(1)}=0 \quad \text{on} \quad S_{lf} \quad \text{and} \quad \delta u_i^{(N)}=0 \quad \text{on} \quad S_{uf} \tag{125}$$

Substituting equation (92) into this equation and making use of the notation (110), one finds

$$\delta J_* = \oint_C \nu_\alpha\, dc \sum_{m=1}^{N} \left\{\left[N_*^{\alpha j} - (V^{\alpha j}+V_0^{\alpha j})\right]\delta v_j + \left[M_*^{\alpha j} - (W^{\alpha j}+W_0^{\alpha j})\right]\delta w_j\right\}^{(m)} \tag{126a}$$

where

$$(N_*^{\alpha j}, M_*^{\alpha j}) = \int_Z \tau_*^{\alpha j}(1, x^3)\,\mu\, dx^3 \tag{126b}$$

and

$$V^{\alpha\beta} = N^{\alpha\beta} - b_\sigma^\beta M^{\alpha\sigma}, \quad V_0^{\alpha\beta} = N_0^{\alpha\beta}(v^\beta_{:\alpha} - b^\beta v_3) + M_0^{\alpha\sigma}(w^\beta_{:\sigma} - b_\sigma^\beta w_3) + Q_0^\alpha w^\beta$$

$$V^{\alpha 3} = Q^\alpha, \quad V_0^{\alpha 3} = Q_0^\alpha w_3 + N_0^{\alpha\beta}(v_{3,\beta} + b_\beta^\sigma v_\sigma) + M_0^{\alpha\beta}(w_{3,\beta} + b_\beta^\sigma w_\sigma)$$

$$W^{\alpha\beta} = M^{\alpha\beta} - b_\sigma^\beta K^{\alpha\sigma}, \quad W_0^{\alpha\beta} = M_0^{\alpha\sigma}(v^\beta_{:\sigma} - b_\sigma^\beta v_3) + K_0^{\alpha\sigma}(w^\beta_{:\sigma} - b_\sigma^\beta w_3) + R_0^\alpha w^\beta$$

$$W^{\alpha 3} = R^\alpha, \quad W_0^{\alpha 3} = R_0^\alpha w_3 + M_0^{\alpha\beta}(v_{3,\beta} + b_\beta^\sigma v_\sigma) + K_0^{\alpha\beta}(w_{3,\beta} + b_\beta^\sigma w_\sigma) \tag{127}$$

After considering the expansions (92b) in (125) and then following some rearrangement of terms, the variational

integral (125) takes the form

$$\delta J_{\star} = \oint_C \nu_\alpha dc \sum_{m=1}^{N} \sum_{i=1}^{3} \sum_{s,t=0}^{P,Q} \Big\langle \{ N^{\alpha(i)}_{\star st} - \sum_{p,q=0}^{P,Q} \big[(\underset{\sim}{V})^{\alpha(i)st}_{pqk} (\underset{\sim}{X})^{k}_{pq}$$

$$+ (\underset{\sim}{W}_o)^{\alpha(i)st}_{pqk} (\underset{\sim}{X}_u)^{k}_{pq} \big] \} \delta\alpha^{st}_{i} + \{ M^{\alpha(i)}_{\star st} - \sum_{p,q=0}^{P,Q} \big[(\underset{\sim}{W})^{\alpha(i)st}_{pqk} (\underset{\sim}{X})^{k}_{pq}$$

$$+ (\underset{\sim}{W}_o)^{\alpha(i)st}_{pqk} (\underset{\sim}{X}_u)^{k}_{pq} \big] \} \delta\beta^{st}_{i} \Big\rangle^{(m)} \tag{128}$$

where the prescribed traction resultants by

$$(N_\star, M_\star)^{\alpha(i)}_{pq} = (T^{\alpha i}_{\star o} - b^{\sigma}_{\beta} \delta^{i}_{\sigma} T^{\alpha\beta}_{\star 1})(v^{pq}_{i}, w^{pq}_{i}) \tag{129}$$

in the notation of (71), the matrices of tractions by

$$(\underset{\sim}{V})^{\alpha(i)st}_{pqk} = (a^{\alpha i}_{pqk} \; a^{\alpha i}_{pqk} \; a^{\alpha i}_{pqk} \; b^{\alpha i}_{pqk} \; b^{i}_{pqk} \; b^{i}_{pq}) v^{st}_{(i)}$$

$$(\underset{\sim}{W})^{\alpha(i)st}_{pqk} = (c^{\alpha i}_{pqk} \; c^{\alpha i}_{pqk} \; d^{\alpha i}_{pqk} \; d^{\alpha i}_{pq}) w^{st}_{(i)} \tag{130}$$

with the coefficients of the form

$$(\underset{\sim}{a})^{\alpha\sigma}_{pqk} = \sum_{r=0}^{2} (\underset{\sim}{\mu}_r) (\underset{\sim}{v})^{\alpha\sigma}_{pqk}, \quad (\underset{\sim}{c})^{\alpha\sigma}_{pqk} = \sum_{r=0}^{2} (\underset{\sim}{\mu}_{r+1}) (\underset{\sim}{v})^{\alpha\sigma}_{pqk}$$

$$(\underset{\sim}{a})^{\alpha 3}_{pqk} = \mu_r (\underset{\sim}{w})^{\alpha 3}_{pqk}, \quad (\underset{\sim}{c})^{\alpha 3}_{pqk} = \mu_{r+1} (\underset{\sim}{w})^{\alpha 3}_{pqk} \tag{131}$$

where

$$(\underset{\sim}{a})^{\alpha\sigma}_{pqk} = (a^{\alpha\sigma}_{pqk} \; a^{\alpha\sigma}_{pq} \; b^{\alpha\sigma}_{pqk} \; b^{\alpha\sigma}_{pq}), \quad (\underset{\sim}{c})^{\alpha\sigma}_{pqk} = (c^{\alpha\sigma}_{pqk} \; c^{\alpha\sigma}_{pq} \; d^{\alpha\sigma}_{pqk} \; d^{\alpha\sigma}_{pq})$$

$$(\underset{\sim}{a})^{\alpha 3}_{pqk} = (a^{\alpha 3}_{pqk} \; a^{\alpha 3}_{pq} \; b^{\alpha 3}_{pqk} \; b^{\alpha 3}_{pq}), \quad (\underset{\sim}{c})^{\alpha 3}_{pqk} = (c^{\alpha 3}_{pqk} \; c^{\alpha 3}_{pq} \; d^{\alpha 3}_{pqk} \; d^{\alpha 3}_{pq})$$

$$(\underset{\sim}{\mu}_r) = (\mu_r \; \mu_{r+1}) \tag{132}$$

and

$$(\underset{\sim}{v})^{\alpha\sigma}_{pqk}=\begin{pmatrix} r^{v\,\alpha\sigma}_{pqk} & r^{\lambda\,\alpha\sigma}_{pqk} & r^{w\,\alpha\sigma}_{pqk} & r^{\xi\,\alpha\sigma}_{pqk} \\ -r^{v\,\alpha\beta}_{pqk}\, b^{\sigma}_{\beta} & -r^{\lambda\,\alpha\sigma}_{pqk}\, b^{\sigma}_{\beta} & -r^{w\,\alpha\beta}_{pqk}\, b^{\sigma}_{\beta} & -r^{\xi\,\alpha\beta}_{pq}\, b^{\sigma}_{\beta} \end{pmatrix}$$

$$(\underset{\sim}{w})^{\alpha 3}_{pqk}=(r^{v\,\alpha 3}_{pqk}\;\; r^{\lambda\,\alpha 3}_{pq}\;\; r^{w\,\alpha 3}_{pqk}\;\; r^{\xi\,\alpha 3}_{pq}) \qquad (133)$$

and the matrices of initial tractions by

$$(\underset{\sim}{V}_o)^{\alpha(i)st}_{pqk}=(a^{\alpha(i)}_{opqk}\;\; b^{\alpha(i)}_{opqk})\cdot{}^{st}_{i}\;,\quad (\underset{\sim}{W}_o)^{\alpha(i)st}_{pqk}=(c^{\alpha(i)}_{opqk}\;\; d^{\alpha(i)}_{opqk})w^{st}_{i}$$

$$(134)$$

with the coefficients of the form

$$a^{\alpha\beta}_{opq(\beta)}=N^{\alpha\beta}_{o}v^{\beta}_{pq:\sigma},\; a^{\alpha\beta}_{opq3}=-N^{\alpha\sigma}_{o}b^{\beta}_{\sigma}v^{pq}_{3},$$

$$a^{\alpha 3}_{opq\sigma}=N^{\alpha\beta}_{o}b^{\sigma}_{\beta}\; v^{pq}_{(\sigma)},\; a^{\alpha 3}_{opq3}=N^{\alpha\beta}_{o}v^{pq}_{3,\beta}$$

$$b^{\alpha\beta}_{opq(\beta)}=M^{\alpha\sigma}_{o}w_{pq:\sigma}+Q^{\alpha}_{o}w^{\beta}_{pq},\; b^{\alpha\beta}_{opq3}=-M^{\alpha\sigma}_{o}b^{\beta}_{\sigma}w^{pq}_{3},$$

$$b^{\alpha 3}_{opq\sigma}=M^{\alpha 3}_{o}b^{\sigma}_{\beta}w^{pq}_{(\sigma)},\; b^{\alpha 3}_{opq3}=M^{\alpha\beta}_{o}w^{pq}_{3,\beta} \qquad (135)$$

$$c^{\alpha\beta}_{opq(\beta)}=M^{\alpha\sigma}_{o}v^{\beta}_{pq:\sigma},\; c^{\alpha\beta}_{opq3}=-M^{\alpha\sigma}_{o}b^{\beta}_{\sigma}v^{pq}_{3},$$

$$c^{\alpha 3}_{opq\sigma}=M^{\alpha\beta}_{o}b^{\alpha}_{\beta}\; v^{pq}_{(\sigma)},\; c^{\alpha 3}_{opq3}=M^{\alpha\beta}_{o}v^{pq}_{3,\beta}$$

$$d^{\alpha\beta}_{opq(\beta)}=K^{\alpha\sigma}_{o}w^{\beta}_{pq:\sigma}+R^{\alpha}_{o}w^{\beta}_{pq},\; d^{\alpha\beta}_{opq3}=-K^{\alpha\sigma}_{o}b^{\beta}_{\sigma}w^{pq}_{3},$$

$$d^{\alpha 3}_{opq\sigma}=K^{\alpha\beta}_{o}b^{\sigma}_{\beta}\; w^{pq}_{(\sigma)},\; d^{\alpha 3}_{opq3}=K^{\alpha\beta}_{o}\; w^{pq}_{3,\beta}$$

are defined.

M a c r o s c o p i c c h a r g e e q u a t i o n of e l e c t r o s t a t i c s

Likewise, after inserting (93) into (16), the variational integral of charge equations by

$$\delta I = \sum_{m=2}^{N-1} \left[\int_A dA \int_Z D^i_{;i}\,\delta(\kappa + x^3\xi)\,\mu dx^3\right]^{(m)} \tag{136}$$

is expressed for the piezoelectric laminae. From this variational integral, in view of the expansion (93b) and after carrying out integrations with respect to the thickness coordinate, one obtains

$$\delta I = \int_A dA \sum_{m=2}^{N-1} \left[(V+l)\,\delta\kappa + (W+m)\,\delta\xi\right]^{(m)} \tag{137}$$

where

$$\begin{gathered} V = C^\alpha_{:\alpha},\quad W = F^\alpha_{:\alpha} - C^3 \;;\; l = c-d,\; m = f-g \\ (C^i, F^i) = \int_Z D^i(1, x^3)\,\mu dx^3 \\ (c,f) = \mu n_3 (1, x^3) D^3 \quad \text{at} \quad x^3 = z+h;\ (d,g) = -\mu n_3 (1,x^3) D^3 \Big|_{x^3 = z-h} \end{gathered} \tag{138}$$

Now, substitution of (93b) into (137) results in

$$\begin{aligned} \delta I = \int_A dA \sum_{m=1}^{N} \sum_{s,t=0}^{P,Q} \sum_{p,q=0}^{P,Q} \Big[& (\underset{\sim}{V}+\underset{\sim}{L})^{st}_{pqk} (\underset{\sim}{X})^k_{pq}\,\delta\gamma_{st} \\ & + (\underset{\sim}{W}+\underset{\sim}{K})^{st}_{pqk} (\underset{\sim}{X})^k_{pqk}\,\delta v_{st} \Big]^{(m)} \end{aligned} \tag{139}$$

where

$$(\underset{\sim}{V})^{st}_{pqk} = (a_{pqk}\ b_{pqk}\ a_{pq}\ b_{pq})\kappa^{st} \quad , \quad (\underset{\sim}{W})^{st}_{pqk} = (c_{pqk}\ d_{pqk}\ c_{pq}\ d_{pq})\nu^{st} \tag{140}$$

with the coefficients of the form

$$\begin{aligned} a_{pqk} &= \sum_{r=0}^{2} {}_r v^\alpha_{pqk:\alpha}\,\mu_r, \quad b_{pqk} = \sum_{r=0}^{2} {}_r w^\alpha_{pqk:\alpha}\,\mu_r, \\ a_{pq} &= \sum_{r=0}^{2} {}_r \kappa^\alpha_{pq:\alpha}\,\mu_r, \quad b_{pq} = \sum_{r=0}^{2} {}_r \xi^\alpha_{pq:\alpha}\,\mu_r \\ c_{pqk} &= \sum_{r=0}^{2} ({}_r v^\alpha_{pqk:\alpha}\,\mu_{r+1} - {}_r v^3_{pqk}\,\mu_r), \\ d_{pqk} &= \sum_{r=0}^{2} ({}_r w^\alpha_{pqk:\alpha}\,\mu_{r+1} - {}_r w^3_{pqk}\,\mu_r) \end{aligned} \tag{141}$$

$$c_{pq}=\sum_{r=0}^{2}({}_r\kappa^{\alpha}_{pq:\alpha}\mu_{r-1}-{}_r\kappa^{3}_{pq}\mu_r),\quad d_{pq}=\sum_{r=0}^{2}({}_r\xi^{\alpha}_{pq:\alpha}\mu_{r-1}-{}_r\xi^{3}_{pq}\mu_r)$$

and

$$(\underset{\sim}{L})^{st}_{pqk}=(\underset{\sim}{c})^{st}_{pqk}-(\underset{\sim}{d})^{st}_{pqk}\quad,\quad(\underset{\sim}{K})^{st}_{pqk}=(\underset{\sim}{f})^{st}_{pqk}-(\underset{\sim}{g})^{st}_{pqk} \tag{142}$$

with the denotations by

$$\left[(\underset{\sim}{c})^{st}_{pqk},(\underset{\sim}{d})^{st}_{pqk}\right]=\kappa^{st}\sum_{r=0}^{2}(\mu'_r,\mu''_r)({}_r\underset{\sim}{v}^3)_{pqk}\ ,$$

$$\left[(\underset{\sim}{f})^{st}_{pqk},(\underset{\sim}{g})^{st}_{pqk}\right]=\xi^{st}\sum_{r=0}^{2}(\mu'_{r+1},\mu''_{r+1})({}_r\underset{\sim}{v}^3)_{pqk} \tag{143}$$

and

$$({}_r\underset{\sim}{v}^i)_{pqk}=({}_rv^i_{pqk}\quad {}_rw^i_{pqk}\quad {}_r\kappa^i_{pq}\quad {}_r\xi^i_{pq}) \tag{144}$$

are introduced

In view of equations (42) and (138), the continuity of surface charge by

$$\varepsilon^{1(m)}_{st}=\sum_{p,q=0}^{P,Q}\left\{\left[(\underset{\sim}{c})^{st}_{pqk}(\underset{\sim}{X})^k_{pq}\right]^{(m)}-\left[(\underset{\sim}{d})^{st}_{pqk}(\underset{\sim}{X})^k_{pq}\right]^{(m+1)}\right\}=0$$

$$m=1,2,\ldots,N-1 \tag{145}$$

$$\varepsilon^{2(m)}_{st}=\sum_{p,q=0}^{P,Q}\left\{\left[(\underset{\sim}{f})^{st}_{pqk}(\underset{\sim}{X})^k_{pq}\right]^{(m)}-\left[(\underset{\sim}{g})^{st}_{pqk}(\underset{\sim}{X})^k_{pq}\right]^{(m+1)}\right\}=0$$

is given.

E l e c t r i c a l   b o u n d a r y   c o n d i t i o n s

As before, the electrical boundary conditions (16) is written in the form

$$\delta I_{\star}=\sum_{m=1}^{N}\oint_C\left\{\int_{Z_m}(\sigma_{\star}-\nu_{\alpha}D^{\alpha})(\delta\kappa+x^3\delta\xi)\mu\, dc\, dx^3\right\}^{(m)} \tag{146}$$

with

$$\delta\phi'=\delta\phi''=0\quad\text{on}\quad S_f \tag{147}$$

for the piezoelectric strained laminae. In the boundary conditions above, the surface charges are taken to be prescribed on the edge boundary surface, while the electric potential is applied to the faces as in equations (82)-(84).

After inserting (93) into (146) and then integrating it with respect to the thickness coordinate, the variational boundary integral can be put in the form

$$\delta I_* = \sum_{m=1}^{N} \oint_C \{[(C_* - \nu_\alpha C^\alpha)\delta\kappa + (F_* - \nu_\alpha F^\alpha)\delta\xi]dc\}^{(m)} \tag{148}$$

where the edge-surface resultants are defined by

$$(C_*, F_*) = \int_Z \sigma_*(1, x^3)\mu dx^3 \tag{149}$$

Lastly, a substitution of the expansion (93b) into the above variational integral gives

$$\delta I_* = \oint_C dc \sum_{m=1}^{N} \Big\langle \sum_{s,t=0}^{P,Q} \{[C_{*st} - \sum_{p,q=0}^{P,Q} (\underset{\sim}{C})^{st}_{pqk} (\underset{\sim}{X})^{k}_{pq}]\delta\gamma_{st}$$

$$+ [F_{*st} - \sum_{p,q=0}^{P,Q} (\underset{\sim}{F})^{st}_{pqk} (\underset{\sim}{X})^{k}_{pq}]\delta\nu_{st}\}\Big\rangle^{(m)} \tag{150}$$

Here, the denotations of the form

$$[C_{*st}, (\underset{\sim}{C})^{st}_{pqk}] = [C_*, (\underset{\sim}{C})_{pqk}]\kappa^{st},$$

$$[F_{*st}, (\underset{\sim}{F})^{st}_{pqk}] = [F_*, (\underset{\sim}{F})_{pqk}]\xi^{st} \tag{151}$$

$$(\underset{\sim}{C}, \underset{\sim}{F})_{pqk} = \nu_\alpha \sum_{r=0}^{2} (\mu_r, \mu_{r+1}) \ ({}_r\underset{\sim}{V}^\alpha)_{pqk}$$

are introduced.

Governing equations of piezoelectric laminae

Now, setting the variational integrals (112), (128), (139) and (150) equal to zero, for the arbitrary and independent variations of the coefficients $(\alpha^1_{st}, \beta^1_{st}, \gamma_{st}$ and $\nu_{st})$, one reads the macroscopic equations of incremental motion as

$$\varepsilon^{(i)st}_{(m)1} = \sum_{p,q=0}^{P,Q} \left[(\underset{\sim}{V}+\underset{\sim}{L})^{(i)st}_{pqk} (\underset{\sim}{X})^{k}_{pq} + (\underset{\sim}{V}_o + \underset{\sim}{L}_o - \rho\,\omega^2 \underset{\sim}{A})^{(i)st}_{pqk} (\underset{\sim}{X}_u)^{k}_{pq} \right]^{(m)}$$

on A (152)

$$\varepsilon^{(i)st}_{(m)2} = \sum_{p,q=0}^{P,Q} \left[(\underset{\sim}{W}+\underset{\sim}{K})^{(i)st}_{pqk} (\underset{\sim}{X})^{k}_{pq} + (\underset{\sim}{W}_o + \underset{\sim}{L}_o - \rho\,\omega^2 \underset{\sim}{B})^{(i)st}_{pqk} (\underset{\sim}{X}_u)^{k}_{pq} \right]^{(m)}$$

with the following natural mechanical edge conditions

$$\varepsilon^{(i)st}_{\star(m)1} = \nu_\alpha \left\{ N^{\alpha(i)}_{\star st} - \sum_{p,q=0}^{P,Q} \left[(\underset{\sim}{V})^{\alpha(i)st}_{pqk} (\underset{\sim}{X})^{k}_{pq} + (\underset{\sim}{W}_o)^{\alpha(i)st}_{pqk} (\underset{\sim}{X}_u)^{k}_{pq} \right] \right\}^{(m)}$$

along C (153)

$$\varepsilon^{(i)st}_{\star(m)2} = \nu_\alpha \left\{ M^{\alpha(i)}_{\star st} - \sum_{p,q=0}^{P,Q} \left[(\underset{\sim}{W})^{\alpha(i)st}_{pqk} (\underset{\sim}{X})^{k}_{pq} + (\underset{\sim}{W}_o)^{\alpha(i)st}_{pqk} (\underset{\sim}{X}_u)^{k}_{pq} \right] \right\}^{(m)}$$

and the macroscopic charge equations of electrostatics as

$$\varepsilon^{(st)}_{(m)1} = \sum_{p,q=0}^{P,Q} \left[(\underset{\sim}{V}+\underset{\sim}{L})^{st}_{pqk} (\underset{\sim}{X})^{k}_{pq} \right]^{(m)}$$

on A (154)

$$\varepsilon^{(st)}_{(m)2} = \sum_{p,q=0}^{P,Q} \left[(\underset{\sim}{W}+\underset{\sim}{K})^{st}_{pqk} (\underset{\sim}{X})^{k}_{pq} \right]^{(m)}$$

with the following natural electrical boundary conditions

$$\varepsilon^{(st)}_{\star(m)1} = C_{\star st} - \sum_{p,q=0}^{P,Q} (\underset{\sim}{C})^{st}_{pqk} (\underset{\sim}{X})^{k}_{pq}$$

along C (155)

$$\varepsilon^{(st)}_{\star(m)2} = F_{\star st} - \sum_{p,q=0}^{P,Q} (\underset{\sim}{F})^{st}_{pqk} (\underset{\sim}{X})^{k}_{pq}$$

Here, it should be noted that $(\varepsilon,\varepsilon_\star)^{(i)st}_{m\alpha}$ and $(\varepsilon,\varepsilon_\star)^{st}_{m\alpha}$ are not equal to zero due to the approximate nature of the expansions (92) and (93). The macroscopic equations above together with the equations (94),(95),(123) and (145) for the continuity of mechanical displacements, tractions, electric potential and surface charge constitute the approximate, higher order governing equations for the piezoelectric strained laminae.

Moment equations

At this final stage, in view of equations (94),(95), (123), (145) and (152)-(155), the moment equations of the form

$$\int_A \left[\varepsilon^{(i)}_{(st)\sigma} + \lambda^{i(st)}_{(\sigma)m}\, \varepsilon^{\sigma(m)}_{(i)st} + \lambda^{(i)}_{m}\, \varepsilon^{(m)}_{i}\right] x_1^s x_2^t\, dA + \oint_C \varepsilon^{\sigma(m)}_{*(i)st}\, \mu^{i(st)}_{(\sigma)m}\, x_1^s x_2^t\, dc = 0 \tag{156}$$

$$\int_A \left[\varepsilon^{m}_{(st)\sigma} + \lambda^{(st)}_{(\sigma)m}\, \varepsilon^{\sigma(m)}_{(i)st} + \lambda_m\, \varepsilon^{(m)}\right] x_1^s x_2^t\, dA + \oint_C \varepsilon^{\sigma(m)}_{*st}\, \mu^{(st)}_{(\sigma)m}\, x_1^s x_2^t\, dc = 0 \tag{157}$$

are established so as to compute the matrix of unknown coefficients as

$$(\underset{\sim}{X})^{k(m)}_{pq} = (\alpha^{k(m)}_{pq}\ \beta^{k(m)}_{pq}\ \gamma^{(m)}_{pq}\ \ \nu^{(m)}_{pq})^T \tag{158}$$

with

$$k=1,2,3\ ;\ (p,s)=1,2,\ldots,P\ ;\ (q,t)=1,2,\ldots,Q\ ;\quad m=2,3,\ldots,N-1 \tag{159}$$

for the governing equations of piezoelectric strained laminae. Here, (m) does not take the values (1) and (N) since the mechanical displacements and the electric potential are given on the upper and lower faces of piezoelectric laminae as already indicated in equations (125) and (147). Also, $\lambda^{i(st)}_{(\sigma)m}$, $\lambda^{(i)}_{m}$, $\lambda^{(st)}_{(\sigma)m}$, λ_m, $\mu^{i(st)}_{(\sigma)m}$ and $\mu^{(st)}_{(\sigma)m}$ are Lagrange undetermined multipliers and are introduced to take into account of the continuity of tractions (123) and mechanical displacements (94), that of surface charge (145) and electric potential (95) and the mechanical and electrical boundary conditions (153) and (155), respectively.

The moment equations (156) and (157) form a system of simultaneous nonlinear algebraic equations in terms of the unknown constants (158), and they can be solved by standard techniques of numerical computation for any special case under consideration.

Some applications of the moment equations for special motions of the piezoelectric strained laminae are the topics of future study, and they will be reported elsewhere.

6- ON SPECIAL CASES

To predict the dynamic response of a piezoelectric strained laminae, two unified algorithms of solutions based upon Kantorovich's method and the method of moments are described in invariant form in the previous two sections. Thus, they can be readily applicable to the macromechanical analysis of the piezoelectric laminae using the most suitable system of coordinates for its geometrical configuration. Now, the results of two unified algorithms are specialized so as to obtain those involving special geometry , kinematics, incremental motion, material properties and mechanical bias.

On geometry

In the absence of curvature effects in which case the shifters are reduced to the Kronecker deltas, namely,

$$b^{\alpha}_{\beta} = 0 \quad ; \quad \mu^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} \tag{160}$$

then the results reduce to those of piezoelectric strained laminae with plane constituents. Likewise, the case of shallow constituents can be readily introduced. Moreover, when a particular geometrical configuration is considered, the results of this particular case can be stated with the help of the succint notation of tensor analysis used herein and by choosing the most appropriate coordinate system for its geometry. As a particular case of interest, consider a piezoelectric laminae with constituents in cylindrical shape. The piezoelectric laminae is referred to the cylindrical coordinate system (x,θ,z) with $x=x^1$ being taken in the axial direction, $\theta=x^2$ in the circumferential direction, $z=x^3$ in the radial direction. The origin of the cylindrical coordinate system is located on the midsurface A of first constituent with its radius of curvature R, and hence, the first, second and third fundamental forms of the midsurface A are recorded in the form

$$a_{11}=R^2=1/a^{11} \ , \ a_{22}=a^{22}=1 \ ; \ b^1_1=-1/R \ ; \ c_{11}=1$$

$$\text{all other } (a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta})=0 \tag{161}$$

Further, when N is taken to be equal to 1, the resulting equations become especially suitable for numerical solutions of a recent theory of piezoceramic shells under a bias [10].

On kinematics

In the field of mechanical displacements (92), by choosing the components as

$$w_\alpha = -(v_{3,\alpha} + b^\sigma_\alpha v_\sigma) \quad , \quad w_3 = 0 \quad ; \quad \xi = 0 \tag{162a}$$

one obtains the Kirchhoff-Love theory of curvilinear piezoelectric strained laminae. By virtue of the continuity of mechanical displacements (41) and that of electric potential (43), equation (162a) implies

$$\bar{u}_\alpha = v_\alpha - x^3(v_{3,\alpha} + b^\sigma_\alpha v_\sigma), \ \bar{u}_3 = v_3 \quad \text{and} \quad \phi = \kappa \tag{162b}$$

for all constituents. Apparently, equation (162) leads to the results of piezoelectric laminae within the frame of the effective modulus theory of composites (e.g., [38]). This has the contradictions of the Kirchhoff-Love hypotheses in each constituent and also it is unable to account for the dynamic interactions at the interfaces of adjacent constituents. Further, the results can be simplified for the case when some of the layers are very thin. Hence, one reads $w_k^{(m)}=0$ and $t^{\alpha 3}_{(m)}=0$ for the very thin (m)-th constituent of piezoelectric laminae. Accordingly, the terms involving with $\delta w_k^{(m)}$ are discarded in the macroscopic equations of incremental motion and the associated mechanical boundary conditions.

On incremental motion

In the field of mechanical displacements (92), v_α stands for the extensional (or the stretching) motions, v_3 and w_α for the flexural (bending) motions, and w_3 for the thickness stretching of constituents. Accordingly, for the flexural type of incremental motions, only the terms involving v_3 and w_α should be kept in the resulting equations of piezoelectric laminae. However, all the terms should be included for the coupled type of incremental motions.

On material properties

Special classes of materials for the constituents of piezoelectric laminae may be considered in the constitutive equations (48), (50), (100) and (101). Of the classes, the constitutive relations of the form

$$C^{ijkl}=[C^{11(i)(i)},C^{i3(i)3}=C^{1313}=\frac{1}{2}(C^{1111}-C^{2222})],$$

$$C^{ijk}=(C^{112},C^{113},C^{333}),\ C^{ij}=C^{(i)i} \tag{163}$$

are recorded for the constituents when the direction of polarization coincides with the thickness coordinate x^3, [39]. In this case, the number of independent material constants is reduced from 45 to 12 as indicated in equation (163).

Mechanical bias

In the present analysis, the piezoelectric laminae is subjected to a general state of initial stresses. Several restrictions can be readily taken up in this mechanical bias. Besides, when the terms involving the bias, that is, th--e indicated by a zero index are discarded, the resulting e ions provide a standard basis for generating approximate -t solutions for the piezoelectric unstrained laminae ., [5]). On the other hand, if the terms involving incremental motions are omitted, the standard basis is provided for a piezoelectric unstrained laminae subjected to large displacement gradients and large angles of rotation. In addition, dropping out the electrical terms, one readily obtains the standard basis for a multilayer shell.

Lastly, a complete linearization by discarding all the terms of mechanical bias leads to a fully linear suystem of algebraic equations, that is, equation (89) for Kantorovich's method and equations (156) and (157) for the method of moments; their solutions are always at hand.

7- CONCLUSION

To provide a stand--d basis for generating approximate direct solutions for the macromechanical analysis of a piezoelectric laminae under mechanical bias, two unified algorithms are presented which are based on Kantorovich's method and the method of moments. Both the methods, though they are well-known in computational physics, are overlooked in piezoelectricity, and they are first formulated herein, within the author's best knowledge, for the numerical treatment of piezoelectric elements. The numerical algorithms are formulated on the basis of the expansions (39), (92) and (93) which are complete in the closure of piezoelectric strained laminae due to Weierstrass's theorem. The formulation being in

tensor notation, the resulting equations may be expressed in any particular coordinate system most suitable for the geometrical configuration of piezoelectric laminae under consideration. The resulting equations by both the methods (89) and (156), (157) take into account all the significant electrical and mechanical effects in the constituents of piezoelectric laminae, and also, all they maintain the continuity of mechanical displacements, electric potential, tractions and surface charge at the interfaces of constituents. These equations accommodate all the incremental types of extensional, thickness and flexural as well as coupled, small motions of a piezoelectric laminae under a general state of initial stresses. Further, special cases are pointed out involving geometry, kinematics, material properties, mechanical bias and small incremental motion.

Both the algorithms formulated herein and especially the algorithm based on the method of moments seem to be an efficient and computationally easy method in investigating the dynamic behavior of piezoelectric strained laminae, as it is anticipated by similar algorithms in electromagnetic theory. The algorithms may be readily extended so as to incorporate the biasing of electrical, thermal and even magnetic fields (e.g., [40]) and also to take into account the viscoelastic properties of constituents by replacing their elastic stiffenesses by their corresponding convolution integrals. Besides, both the algorithms may be formulated for the macromechanical analysis of a piezoelectric one-dimensional element [41] .

In closing, the application of two algorithms which remains to be exhibited for certain motions of the piezoelectric laminae by choosing its geometry, electro-mechanical properties of constituents and mechanical bias is a topic of forthcoming studies.

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